Talk 2

Yesterday we have we have studied the deformation theory of

Smooth of dim $4g-4+n$ $\leftarrow$ $Pic^d_{g,n} = \{(C, p_1, \dots p_n; L) \mid \cdot (C, p_1 \dots p_n) \text{ pre-st.}, \cdot \deg L = d\}/\sim$

p = Forget L $\rightarrow$ smooth of dim $g-1$

$\downarrow$

$\mathcal{M}_{g,n} = \{(C, p_1, \dots p_n) \mid \text{pre-stable}\}/\simeq$

$\hookrightarrow$ smooth of dim $3g-3+n$

Finally, to understand dimension and regularity of

$\overline{M}_{g,n}(\mathbb{P}^2, d) = \{(C, \underline{p}, L, s_0 \dots s_r) \mid \rightarrow s_i \text{ define a map}, \rightarrow \text{the map is stable}\}/\sim$

we have to understand the map

$$\boxed{\overline{M}_{g,n}(\mathbb{P}^2, d) \xrightarrow[\varsigma]{} Pic^d_{g,n}}$$

One we have (C, L), in order to give a map to $\mathbb{P}^2$

we have to choose $r+1$ sections of $H^0(C, L)$

$\uparrow$ when C is smooth and $\boxed{d > 2g-2}$

then $H^1(C, L) = 0$

Riemann-Roch on smooth curves } and $\boxed{\dim H^0(C, L) = d-g+1}$

$\Rightarrow$ we would expect ς to have dimension $\boxed{(r+1)(d-g+1)}$

But $\boxed{\text{is } \varsigma \text{ smooth?}}$ i.e given a deformation of (C, L), do the sections $s_0, \dots s_r$ deform?

$\boxed{\text{NO}}$: the $\boxed{\text{obstruction lie in } H^1(C, L)}$

Recall that for example, without moving C,

we had $0 \to L \xrightarrow{\varepsilon} \mathcal{L} \to L \to 0$

$\Rightarrow \dots H^0(\mathcal{L}) \xrightarrow{r} H^0(L) \xrightarrow{S} H^1(L)$

$\qquad\qquad\qquad\qquad s$

if $S(s) = 0 \Rightarrow$ the section extends

otherwise NOT!! [see [Wang, " Def. of pairs (X, L)" with X singular] for formal treatment

NOT smooth
obs in $H^1(C,L)$

smooth of dim $g-1$

$$\overline{M}_{g,n}(\mathbb{P}^2,d) \xrightarrow{S} \mathrm{Pic}^d_{g,n} \xrightarrow{p} \mathcal{M}_{g,n}$$

$\mathrm{Pic}^d_{g,n}$: smooth of dim $4g-4+n$

$\mathcal{M}_{g,n}$: smooth of dim $3g-3+n$

$\overline{M}_{g,n}(\mathbb{P}^2,d)$: it should have dim

$$\to \quad \underbrace{(r+1)(d-g+1)}_{\text{dim of } S \text{ when } C \text{ is smooth}} + 4g-4+n = \underbrace{(\dim\mathbb{P}^2-3)(1-g) + \overset{-\beta\cdot K_{\mathbb{P}^n}}{\overset{\shortparallel}{d(r+1)}} + n}_{:=\text{vir.dim}}$$

§.4. Irreducible components of $\overline{M}_{1,0}(\mathbb{P}^2,3)$

[Friday we will see how complicated is $\overline{M}_{2,0}(\mathbb{P}^2,4)$]

$H^1(L) \neq 0$

(a) In genus 0 this never happens: $L=f^*\mathcal{O}(1)$ has degree ≥ 0 on each irreducible component of C $\Rightarrow H^1(C,L)=0$ for $g(C)=0$

We have just proved that

$$\boxed{\overline{M}_{0,n}(\mathbb{P}^2,d) \text{ is smooth !}}$$

(b) For $g=1$ we can have $H^1 \neq 0$ when $C=$ [sketch] and $\boxed{L|_E = \mathcal{O}_E}$, i.e. when the genus 1 $H^1(L)=\mathbb{C}$ is contracted [Think back at Example 1]

$\hookrightarrow$ such $[f]$ can be a singular point in $\overline{M}_{1,n}(\mathbb{P}^2,d)$

(c) For $g=2$ we can have two bad cases

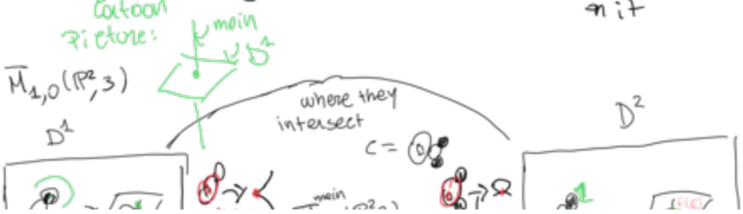
- either $\exists\, Z\subset C \xrightarrow{f} \mathbb{P}^2$ with $g(Z)=1$ or 2 which gets contracted dim $H^1(L)= {1 \atop 2}$
- Or there is $Z\subseteq C$ of genus 2 s.t. $f|_Z : Z\to\mathbb{P}^2$ is the hyperelliptic cover of a line $L\subset\mathbb{P}^2$ [Think back at example 2]

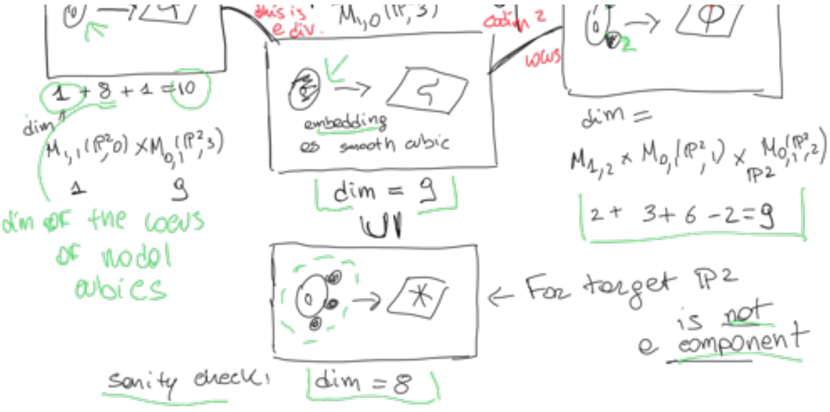
$\Rightarrow H^1(L) = H^1(\omega_Z) = \mathbb{C}$

Convention: [sketch] empty cpt = map contracts it
filled cpt = map has positive deg on it

Cartoon Picture:

$\overline{M}_{1,0}(\mathbb{P}^2,3)$





Theorem

The irreducible components of $\overline{M}_{1,0}(\mathbb{P}^2,d)$ are:

- $\overline{M}_{1,0}^{main}(\mathbb{P}^2,d)$:= closure of the locus where the source curve is smooth.

- $D^k \simeq \overline{M}_{1,k}\times\overline{M}_{0,1}(\mathbb{P}^2,d_1)\times_{\mathbb{P}^2}\overline{M}_{0,1}(\mathbb{P}^2,d_2)\ldots\times_{\mathbb{P}^2}\overline{M}_{0,1}(\mathbb{P}^2,d_k)$

Questions:

1) Can we understand the singularities when 2 components intersect? → local equations for the moduli space
2) Can we characterise the closure of the main component? → Factorization through a non degenerate map

can we give a modular meaning to maps in the intersection of $\overline{M}^{main}$ with D^k

§5. Local equations for $\overline{M}_{1,0}(\mathbb{P}^2,d)$ (and $\overline{M}_{2,0}(\mathbb{P}^2,d)$) and blow-ups

- g=1 work of Vakil-Zinger, Hu-Li 2009-2010
- g=2 " " Hu-Li-Niu, 2018

$$\overline{M}_{1,0}(\mathbb{P}^2,d) \overset{\text{open}}{\underset{\text{non deg. stability}}{\subset}} \mathcal{C}(\pi_*\mathcal{L}_{Pic}^{\oplus r+1}) \quad \text{cone of sections}$$

$\mathcal{C}(\pi_*\mathcal{L}_{Pic}^{\oplus r+1}) = \{(C,L),\ s\in H^0(L^{\oplus r+1})\}$

$\downarrow \mathcal{S}$ $\quad \mathcal{C}\leftarrow\mathcal{L}_{Pic}$

$Pic^d_{1,0} \xleftarrow{\pi} \mathcal{C}$

$L=\mathcal{O}(S_1+S_2+S_3)$

Let us look around a point $[F]=(C,L,\{s_i\})\in\overline{M}_{1,0}(\mathbb{P}^2,d)$

And let $U\to\overline{M}_{1,0}(\mathbb{P}^2,d)$ chart around $[F]$

- we can assume that $\boxed{L = f^*\mathcal{O}(1) = \mathcal{O}(S_1 + \cdots + S_d)}$ $\hookrightarrow$ distinct smooth

we denote by $\mathcal{V} \to \mathrm{Pic}^d_{1,n}$ the chart containg $\boxed{\xi[F]}$

- we can choose on $\mathcal{V}$ divisors A, P, P_0 passing through the genus 1 curve

$= \xi[F]$

The universal line bundle $\mathcal{L}$ on $\mathcal{V}$ is:

$$\boxed{\mathcal{L}|_{\mathcal{V}} = \mathcal{O}_{\mathcal{C}}(D_1 + \cdots + D_d + P - P_0)}$$

- where D_i are extensions of S_i over $\mathcal{V}$
- and you can think of P as a local coordinate on $\mathrm{Pic}^d_{1,n}$, relative to $\mathcal{M}_{1,n}$.
- Moreover on $\mathcal{V}$ we have local coordinates $t_1, \dots t_m$ pulled-back from $p(\mathcal{V}) \to \mathcal{M}_{1,n}$ which are the smoothing parameters of the nodes of $[C] = p(\xi[F])$
- notice that $V = \mathrm{Spec}_{\mathcal{V}}\left(\pi_* \mathcal{L}_{|\mathcal{V}}(A)^{\oplus r+1}\right)$ is a vector bundle on $\mathcal{V}$

$$\begin{array}{c} \mathcal{C}_{\mathcal{V}} \\ \downarrow \pi \\ \mathcal{V} \end{array}$$

$\Rightarrow$ Theorem: There is a $\boxed{\text{closed embedding}}$ of

$$\begin{tikzcd}
\overline{M}_1(\mathbb{P}^r, d) & U \arrow[l] \arrow[r, hook, "i"] \arrow[d] & V \arrow[dl] \\
\mathrm{Pic}^d_{g,n} & \mathcal{V} \arrow[l] &
\end{tikzcd}$$

$\leftarrow$ vector bundle on $\mathcal{V}$

where V is smooth since $\mathcal{V}$ is. Let $\mathcal{C}_{\mathcal{V}} \xrightarrow{\pi} \mathcal{V}$ be the universal curve and universal l.b. over $\mathcal{V}$

$$\begin{array}{c} \mathcal{L}_{\mathcal{V}} \\ \downarrow \\ \mathcal{C}_{\mathcal{V}} \end{array}$$

$\Rightarrow$ we have a s.e.s

$$\boxed{0 \to \pi_* \mathcal{L}^{\oplus r+1} \to \pi_* \mathcal{L}(A)^{\oplus r+1} \xrightarrow{ev_A} \pi_* \mathcal{L}(A)|_A^{\oplus r+1} \to 0}$$

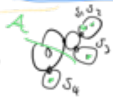
which determines the equations of $U \hookrightarrow V$:

$$\boxed{F_0 = \cdots = F_r = 0}$$

where $F_j = \sum_{i=1}^{d} \prod_k t_k \, w^i_j := \sum_{i=1}^{d} t_{[S_i, A]} w^i_j$

- w^i_j are coordinates on the fibers on the j-th copy of $\pi_* \mathcal{L}|_U(A)^{\oplus r+1}$
- $t_{[S_i, A]}$ is the product of all the smoothing

parameters of the nodes S_i from the component containing A

E.g  $F_J = t_1 \cdot t_2 (w_J^1 + w_J^2) + t_3 t_2 w_J^3 + t_4 w_J^4 = 0$

In genus 2 the idea to find equations is similar, but describing them is more more complicated, as one has to worry about when

nodes  correspond to

Weierstrass conjugate points $\Big\}$ i.e. ramification w.r.t. $Z \xrightarrow{2:1} \mathbb{P}^1$ related by σ_2

$$\begin{array}{ccc} & \longrightarrow & \overline{M}_{1,n}(\mathbb{P}^r, d) \\ \downarrow & \lrcorner & \downarrow \\ \widetilde{M} & \xrightarrow[\text{blowup}]{} & M_{1,n}^{wt} \end{array}$$

Resolution:

It is easier to do the blowup over $M_{1,n}^{wt}$ and then take the fiber product

$M_{1,n} \xleftarrow{\text{finite}} M_{1,n}^{wt\text{-}st} = \{ C + wt : V(\Gamma) \to \mathbb{N} \mid$ (irreducible cpts $= V(\Gamma)$)

(✱) $\left[\begin{array}{l} \cdot\ wt(v) = 0,\ g(v) = 0 \Rightarrow C_v \text{ has } \geq 3 \text{ special pts} \\ \cdot\ wt(v) = 0,\ g(v) = 1 \Rightarrow \quad " \quad \geq 1 \quad " \end{array} \right\}$

$C =$  $\longleftrightarrow$ 

$[C] \in \Theta_k$ looks like 

We consider:

$\Theta_k = \{ C \mid \cdot\ wt(E) = 0$, $E \subset C$ is the minimal $g = 1$; $\cdot\ C$ has at least k nodes $\}$

$\cap|$ codim k

$M_{1,n}^{wt}$

around a generic pt: $t_1 = t_2 = 0$

$\downarrow$

Blow-up successively $\Theta_1, \Theta_2, \dots$

Since we have a stability (✱) this process terminate at $k = d$

$\Rightarrow$ If we look at

$$\begin{array}{ccc} \widetilde{M}_{1,n}(\mathbb{P}^r, d) & \longrightarrow & \overline{M}_{1,n}(\mathbb{P}^r, d) \\ \downarrow & & \downarrow \\ \widetilde{M}^{wt} & & M^{wt} \end{array}$$

$\mathcal{M}^{\cdots}_{1,n} \longrightarrow \mathcal{M}^{\cdots}_{1,n}$

And we look at the proper transforms of the equations $\{F_0 = \dots = F_n\} = 0$ we find equations that looks like

$$\boxed{S = \{ \{ y_0 = \cdots = \{ y_r = 0 \} }$$

And $\widetilde{\mathcal{M}}^{main}_{1,0}(\mathbb{P}^r, d)$ is given by $\boxed{y_0 = \cdots = y_r = 0}$ in this chart.

In genus 2 one one also need to blow-up loci of the form:

Depends on The Brill-Noëther theory of the curve

V_{k,s_1,s_2} - {There are at least k-nodes and 2 of them are conjugated under the involution of the 2:1 cover}

$W_{k,i}$ - {There are at least k-nodes and one of them is a ramification point of the 2:1 cover

↳ The blow up produce $\widetilde{\mathcal{M}}^{main}_{g,n}(\mathbb{P}^r, d)$ $g = 1, 2$ smooth and birational to $\overline{\mathcal{M}}^{main}_{g,n}(\mathbb{P}^r, d)$ ■

The construction is extraordinary! But

- The procedure is very involved and keeping track of the local structure around any possible type of point requires quite some effort!

- The blow up has (apparently) destroyed the modular meaning.

Talk 3

§ 6. Another way to desingularize $\overline{\mathcal{M}}_{g,0}(\mathbb{P}^r, d)$

The obstruction come from $H^1(L) \neq 0$ when $L = f^*\mathcal{O}(1)$

→ f contract a component of genus 1 and 2

(→ f has degree 2 and so the image is a $\mathbb{P}^1$