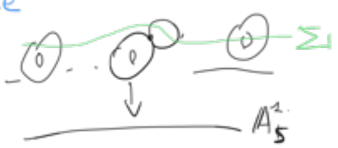
Example / Exercise

$\omega_\pi(D+\Sigma)$ D supp are the contracted locus

Let $\left[\begin{array}{c} \mathcal{C} \\ \downarrow \\ S \end{array}\right] =$



can we find M like we asked before?

Recall:

- ADJUNCTION FORMULA ON SURFACES : $Z \overset{\text{comp.}}{\subseteq} \overset{\text{Fiber}}{C_s}$

$$\omega_{\mathcal{C}/S}|_Z = \omega_Z(-Z^2)$$

- To compute Z^2 recall that for a fiber C_s

$$(C_s)^2 = 0$$

- For a genus 1 nodal curve Z with no rational tails attached $\omega_Z = \mathcal{O}_Z$.

and $\omega_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$

- how do the fibers of the contraction look like

- How do I get a family looking like in the picture?

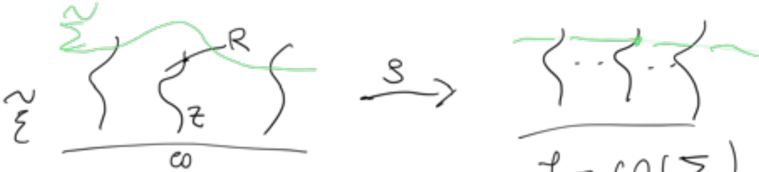
Take your favourite pencil of smooth cubics

$$\mathcal{E} = \{zy^2 - x^3 + sx^2z + z^3\} \subseteq \mathbb{P}^2_{[x:y:z]} \times \mathbb{A}^1_S$$

$$\Sigma = [0:1:0] \times \mathbb{A}^1_S$$

$$p = ([0:1:0], s=0)$$

$$Bl_p\,\mathcal{E} \subseteq \mathbb{P}^2_{[x:y:z]} \times \mathbb{A}^1_S \times \mathbb{P}^2_{[t_0:t_1:t_2]}$$



$$\mathcal{L} = \mathcal{O}_{\mathcal{E}}(\Sigma_1)$$

Who is

$$\rho^*\mathcal{L} = \mathcal{O}(\tilde{\Sigma} + R)$$ exceptional divisor

$$\rho^*\mathcal{L}|_Z = \underset{\overset{\shortparallel}{0}}{\tilde{\Sigma}\cdot Z} + \underset{p}{R\cdot Z}$$

$\rho^*\mathcal{L}(-Z) \leftarrow$ this is trivial on Z and ample everywhere else

$\omega_\pi(eZ)(\tilde\Sigma)$

- away from the central fiber $\mathcal{E}_s$ $s \neq 0$

$$\omega_\pi(eZ)(\tilde\Sigma)\big|_{\mathcal{E}_s} = \underset{\mathcal{O}_{\mathcal{E}_s}}{\underset{\parallel}{\omega_{\mathcal{E}_s}}} + \tilde\Sigma|_{\tilde C} = \mathcal{O}_{\mathcal{E}_s}(p) \checkmark$$

- $\omega_\pi(eZ)(\tilde\Sigma)\,|_Z$

$\hookrightarrow$ adjunction

$$\omega_\pi|_Z + eZ^2 + \overset{0}{\overset{\parallel}{\tilde\Sigma\cdot Z}}$$

$\parallel$ adj.

$\omega_Z - Z^2$

$\parallel$ Z is a smooth $g=1$

$\mathcal{O}_Z$

Take $e=1 \Rightarrow \omega_\pi(Z)(\tilde\Sigma)$ ✓

$$\parallel$$

$$\rho^*\mathcal{L}(-R)$$

$$\parallel$$

$$\omega_\pi(y)(\tilde\Sigma)$$

$$\bar{\mathcal{E}} = \underline{\mathrm{Proj}}_{\mathbb{A}^1_s}\left(\pi_*(\rho^*\mathcal{L}(-R))\right)$$

- How does $\bar{\mathcal{E}}_0$ looks like?

we have to understand the sections of

$$\pi_*\Big(\underbrace{\rho^*\mathcal{O}(k\Sigma)(-kR)}_{\mathcal{O}(k\Sigma)\otimes\tilde{\mathcal{L}}_p^{\otimes k}}\Big)$$

$$\begin{tikzcd} \tilde{\mathcal{E}} \arrow[d, "\rho"] \arrow[dd, bend right, "\pi"'] \\ \mathcal{E} \arrow[d, "\pi'"] \\ \mathbb{A}^1_s \end{tikzcd}$$

understand

$$\pi'_*\left(\mathcal{O}(k\Sigma)\otimes\tilde{\mathcal{L}}_p^{\otimes k}\right)$$

Let's understand the sections of

$$\mathcal{O}_Z(kp) \supset \mathcal{O}_Z((k-1)p) \dots \supset \mathcal{O}_Z(3p) \overset{s_p}{\supset} \mathcal{O}_Z(2p)$$

$$\cup\, s_p$$

$$\mathcal{O}_Z(p)$$

$$\cup\, s_p$$

$$\mathcal{O}_Z$$

s_k, s_{k-2}, s_{k-3}, ..., s_0

vanish of order k in p vanish of order $k-2$ in p

$$H^0(\mathcal{O}_Z(p)) = \mathbb{C}$$
$$H^0(\mathcal{O}_Z(2p)) = \mathbb{C}^2 = \langle s_p^2, s' \rangle_p$$

does not vanish all

since it does not come from $\mathcal{O}_Z(p)$

$\boxed{\text{These section can be extended locally on } \mathbb{A}^1\text{'s}}$

The obstruction to extend section are in

$$H^1(C_0, L)$$

but $k \geq 1$ $H^1(Z, \mathcal{O}_Z(kp)) = 0$

These $\langle s_0, \dots \quad s_{k-2}, \underline{s_k} \rangle \in \pi'_* \mathcal{O}(\Sigma')$
$\mathbb{C}[[s]]$

$\pi'_*(\mathcal{O}(\Sigma') \otimes \mathcal{L}^{\otimes k})$ i.e we wanted all section vanishing to order at least k in p

$$\langle \underline{s_0} \cdot s^k, s_1 \cdot s^{k-1}, \dots \dots \underline{s_{k-2}} \cdot s^2, s_k \rangle_{\mathbb{C}[[s]]}$$
$$\| $$
$$\pi'_* \mathcal{O}(k\Sigma') \otimes \mathcal{L}^{\otimes k}$$
$$= \pi_* \mathcal{O}(k\tilde{\Sigma})(-kR)$$

we can think of these as sect on the blowup

$$\frac{s_0 s^k}{s_R^k}, \frac{s_1 s^{k-1}}{s_R^k}, \dots \frac{s_{k-2} s^2}{s_R^k}, \frac{s_k}{s_R^k}$$

$\boxed{s_Z \cdot s_R = s}$

$s_Z = 0$

$s_R = 0$ R

$$s_0 s_Z^k, \frac{s_1}{s_R} s_Z^{k-1}, \dots \frac{s_{k-3}}{s_R^{k-3}} s_Z^3,$$
$$\frac{s_{k-2}}{s_R^{k-2}} s_Z^2$$
$$\frac{s_k}{s_R^k}$$

constant coefficient

is a local coordinat on R

$s_Z := t$

$\boxed{\alpha_0 t^k, \alpha_1 t^{k-1} \dots \alpha_{k-3} t^3, \alpha_{k-2} t^2, \alpha_k}$

This is the parametrization

of a ωsp 

you see it because there is no linear term.

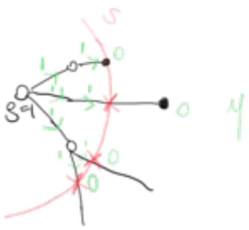
$\bar{e} =$ 

 $\to$ α can never be smoothed

Exercise

- Using the l.b. description of Lecture 4, determine the value of y on each vertex of the curve above
- Prove that $u|_{ev} = 0$ for each v in the strict interior of the circle and has deg > 0 otherwise

$$\boxed{y = \max\{s-\lambda, 0\}}$$



o = wt 0

• = wt > 0