Bridgeland stability conditions and geometric applications.

- Derived categories + bounded t-structures.
- Stability conditions on triangulated categories.
- " " on curves, surfaces (K3 surfaces)
- Applications of in algebraic geometry.

μ - stability on curves.

C: smooth projective curve $/ \mathbb{C}$

$E \in coh(C) \rightsquigarrow \mu(E) := \begin{cases} \deg(E)/rk(E) & \text{if } rk(E) \neq 0 \\ +\infty & \text{other.} \end{cases}$

<u>Def</u>. $E \in coh(C)$ is μ-(semi) stable if $0 \neq F \subset E$

$\mu(F) < (\leq) \mu(E)$.

<u>Ex. 1</u> see-saw property. $0 \to A \to B \to C \to 0$ in $coh(C)$

$\mu(A) < \mu(B) \iff \mu(B) < \mu(C)$.

<u>Ex. 2</u>. take μ-semistable sheaves E, E' s.t. $\mu(E) > \mu(E')$

then $Hom(E, E') =$

Thm. $F \in coh(C)$, there is a unique filtration $0 = F_0 \subset F_1 \subset \cdots \subset F_n = F$

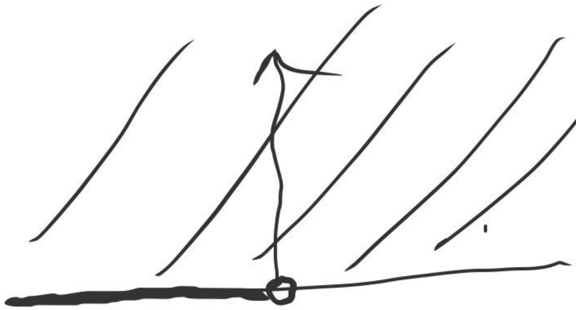
s.t F_i / F_{i-1} are μ-semistable and

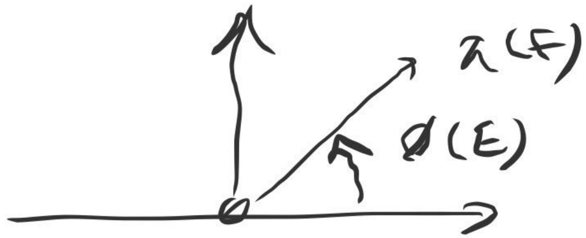
$$\mu(F_1/F_0) > \mu(F_2/F_1) > \cdots$$

$E \in coh(C) \rightsquigarrow Z(E) = i\, rk(E) - deg(E)$

$= r(E) \exp(i\pi \phi(E))$

$0 < \phi(E) \leq 1$





E is μ-(semi) stable if $0 \neq F \subset E$ $\phi(F) < (\leq)\ \phi(E)$.

$$Z : K(C) \longrightarrow \mathbb{C}$$
$$\| $$
$$K(coh(C))$$

K-group of an abelian category is the quotient of free abelian group generated by its objects by $[B] = [A] + [C]$ for any short exact sequence $0 \to A \to B \to C \to 0$

Derived categories and t-structures.

$\mathcal{A}$: abelian category s.t $\mathcal{A}$ has enough injectives.

$\rightsquigarrow C^b(\mathcal{A})$: objects: bounded complexes

$$E^\bullet = \cdots \to E^i \xrightarrow{d^i} E^{i+1} \xrightarrow{d^{i+1}} E^{i+2} \to \cdots$$

with $d^{i+1} \circ d^i = 0$ and $H^i(E^\bullet) = \frac{\ker d^i}{\operatorname{Im} d^{i-1}} = 0$

for all but finitely many i.

$f : E^\bullet \to F^\bullet \rightsquigarrow f^i : E^i \to F^i$

that commute with the differentials

$\underline{K^b(\mathcal{A})}$: objects are the same as $C^b(\mathcal{A})$.

two morphisms $f, g : E^\cdot \to F^\cdot$ are identical if

$f^\cdot - g^\cdot =: h$ is homotopic to zero, i.e.

$$\begin{tikzcd}
\cdots \arrow[r] & E^{i-1} \arrow[r] \arrow[d, "h^{i+1}"] & E^{i} \arrow[r] \arrow[d, "h^{i}"] \arrow[dl, "k^{i}"'] & E^{i+1} \arrow[r] \arrow[d, "h^{i+1}"] \arrow[dl, "k^{i+1}"'] & \cdots \\
\cdots \arrow[r] & F^{i-1} \arrow[r, shift left] \arrow[r, shift right] & F^{i} \arrow[r] & F^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

$$(f-g)^i = h^i = d \circ k^i - k^{i+1} \circ d$$

$\rightsquigarrow D^b(\mathcal{A})$: obtained from $K^b(\mathcal{A})$ by inverting quasi-isomorphisms

(a morphism $f : E^\cdot \to F^\cdot$ which induces isomorphism $H^i(f) : H^i(E^\cdot) \to H^i(F^\cdot)$

for all i).

$$\begin{tikzcd}
& G \arrow[dl, "f"', "\text{q.is}"] \arrow[dr, "g"] & \\
E^\cdot & & F^\cdot
\end{tikzcd} \qquad (g \circ f^{-1})$$

$D^b(\mathcal{A})$ is a triangulated category, satisfying axiom TR1-4.

- shift functor $[1] : D^b(A) \to D^b(A)$ $\qquad (E[1])^i = E^{i+1}$

  $E^\cdot \to \underline{E^\cdot[1]}$ $\qquad d^i_{E[1]} = -d^{i+1}_E$

- cone : any morphism $f : A^\cdot \to B^\cdot \rightsquigarrow \mathrm{cone}(f) \in D^b(\mathcal{A})$

  $A^\cdot \to B^\cdot \to \mathrm{con}(f) \to A[1]$ is an exact triangle.

<u>Ex. 3</u>. $F, G \in \mathcal{A}$ $\qquad \mathrm{Hom}_{D^b(A)}(F[p], G[q]) = \begin{cases} 0 & \text{if } p > q \\ \mathrm{Ext}^{q-p}(F, G) & \text{if } p \leq q \end{cases}$

* All information of $E^\cdot \in D^b(A)$ is not encoded in $H^i(E^\cdot)$

- $\mathcal{A} \subset D^b(\mathcal{A})$ is a heart
- a bound t-structure is uniquely determined by its heart.

<u>prop</u>. the heart $\mathcal{A}^\#$ is abelian.

Exact sequence: exact triangle $A \to B \to C$ in $\mathcal{D}$
s.t. $A, B, C \in \mathcal{A}^\#$.

<u>eg</u>. $f: A \to B \in \mathcal{A}^\#$ is inclusion if $\mathrm{cone}(f) \in \mathcal{A}^\#$

is surjective if $\mathrm{cone}(f) \in \mathcal{A}^\#[1]$.

<u>Ex. 4</u>. Any exact triangle $A \to B \to C$ induces a long-exact cohomology sequence with cohomology objects $H^i_\#(-)$.

$\underline{\underline{\mathcal{A}}} \rightsquigarrow \underline{\underline{D^b(\mathcal{A})}}$

filtration by cohomologies: $E^\bullet \in D^b(A)$,

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & E^\bullet_k \arrow[r] \arrow[d] & E^\bullet_{k-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & E^\bullet_{j+1} \arrow[r] & E^\bullet_j = E^\bullet \arrow[d] \\
& H^{-k}(E)[k] \arrow[ul, dashed] & H^{-k+1}(E^\bullet)[k-1] \arrow[ul, dashed] & & & H^{-j}(E)[j] \arrow[ul, dashed]
\end{tikzcd}$$

heart of a bounded t-structure in a triangulated category D

is a full additive subcategory $A^\# \subset D$ s.t.

(1) $k_1 > k_2$, $\mathrm{Hom}(A^\#[k_1], A^\#[k_2]) = 0$

(2) $E \in D$, $\exists\ k_1 > k_2 > \dots > k_n$ s.t. a sequence of exact triangles.

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_{n-1} \arrow[r] & E = E \arrow[d] \\
& A_1 \arrow[ul, dashed] & A_2 \arrow[ul, dashed] & & & A_n \arrow[ul, dashed]
\end{tikzcd}$$

$A_i \in A^\#[k_i]$

cohomology objects $H^{-k_i}_{A^\#}(E)$ w.r.t $A^\#$

tilting at a torsion pair:

A torsion pair in an abelian category $\mathcal{A}$ is $(\mathcal{T}, \mathcal{F})$ of full additive subcategories wide

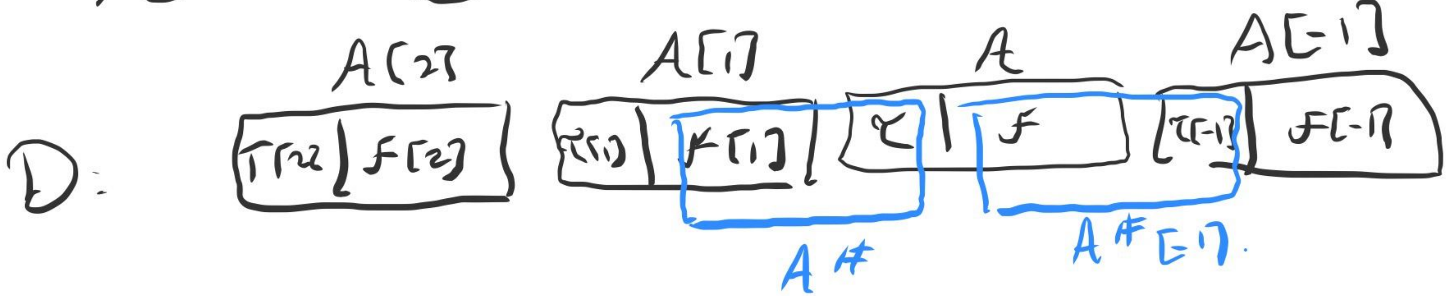
(1) $Hom(\mathcal{T}, \mathcal{F}) = 0$

(2) $E \in \mathcal{A}$, $\exists$ short exact sequence $0 \to T \to E \to F \to 0$ in $\mathcal{A}$

$T \in \mathcal{T}$, $F \in \mathcal{F}$.

Example. $\mathcal{A} = Coh(X) \to \begin{cases} \mathcal{T} = \{ \text{torsion-sheaves on } X \} \\ \mathcal{F} = \{ \text{torsion-free sheaves on } X \}. \end{cases}$

$\mathcal{A} \subset \mathcal{D}$: heart of a bounded t-structure



<u>Prop</u>. Given a torsion-pair $(\mathcal{T}, \mathcal{F})$ in the heart $\underline{\mathcal{A}} \subset \mathcal{D}$ of a bounded t-structure, then the following defines another heart of a bounded t-structure.

$$\mathcal{A}^\# := \langle \mathcal{F}[1], \mathcal{T} \rangle$$
$$= \{ E \in \mathcal{D} : \underline{H}^0(E) \in \mathcal{T},\ \underline{H}^{-1}(E) \in \mathcal{F},\ H^i(E) = 0 \text{ if } i \neq 0, -1 \}$$

Ex. 5. If $\mathcal{A}, \mathcal{A}^\# \subset \mathcal{D}$ are the heart of a bounded t-structure in a triangulated category $\mathcal{D}$ s.t. $\mathcal{A}^\# \subset \langle \mathcal{A}, \mathcal{A}[1] \rangle$ then $\mathcal{A}^\#$ is obtained from $\mathcal{A}$ by a tilt.

Hint: torsion pair: $\mathcal{T} = \mathcal{A} \cap \mathcal{A}^\#$, $\mathcal{F} = \mathcal{A} \cap \mathcal{A}^\#[-1]$.

stability conditions

<u>Def</u> A slicing P of a triangulated category D is a collection of full additive subcategories $P(\phi)$ for each $\phi \in \mathbb{R}$ satisfying:

(1) $P(\phi + 1) = P(\phi)[1]$.

(2) for all $\phi_1 > \phi_2$, $\mathrm{Hom}(P(\phi_1), P(\phi_2)) = 0$

(3) for each $0 \neq E \in D$, there is a sequence $\phi_1 > \phi_2 > \cdots > \phi_n$ of real numbers and a sequence of exact triangles

(*)
$$\begin{tikzcd}[column sep=small]
0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[rr] \arrow[dl] & & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul, dashed] & & A_2 \arrow[ul, dashed] & & & & & A_n \arrow[ul, dashed] &
\end{tikzcd}$$

with $A_i \in P(\phi_i)$

. we call objects in $P(\phi)$ semistable of phase ϕ.

. The filtration (*) is Harder–Narasimhan (HN) filtration of E.