Thm. Let X be a smooth projective surface, then

$$\mathrm{Amp}(X) \times NS(X)_{\mathbb{R}} \longrightarrow \mathrm{Stab}_{\Lambda}(X) \qquad \Lambda = K_{num}(X).$$

$$(\omega, B) \longmapsto \sigma_{\omega,B} = (\mathrm{Coh}^{\omega,B}(X), Z_{\omega,B}).$$

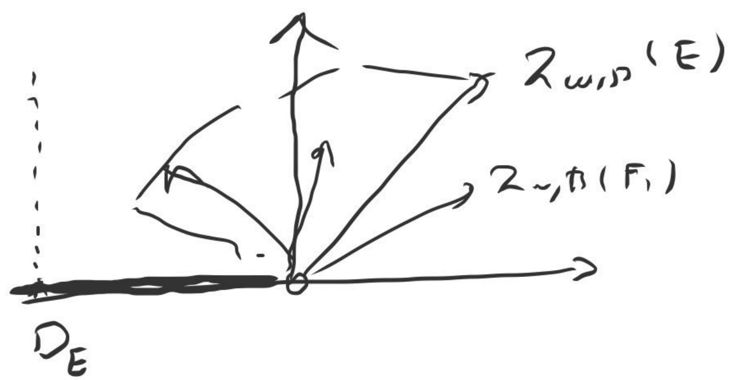
is a continuous embedding.

Pf. ① $Z_{\omega,B}$ is a stability function.

we assume $B, \omega \in NS(X)_{\mathbb{Q}}$

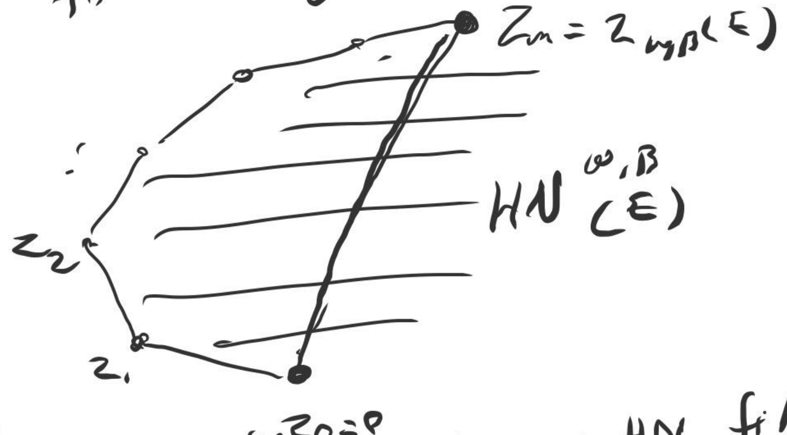
$\rightsquigarrow$ ② $\mathrm{Coh}^{\omega,B}(X)$ is noetherian.

③ Harder-Narasimhan polygon of E : $HN^{\omega,B}(E)$: convex hull of $Z_{\omega,B}(F)$ of $F \subset E$.



Ex. 9. Use the fact $\mathrm{Coh}^{\omega,B}(X)$ is noetherian and the assumption that $\mathrm{Im}[Z_{\omega,B}(-)]$ is discrete in $\mathbb{R}$ to show for any object $E \in \mathrm{Coh}^{\omega,B}(X)$ $\exists\, D_E \in \mathbb{R}$ s.t for any $F \subset E$, $D_E \leq \mathrm{Re}[Z_{\omega,B}(F)]$.

$\Rightarrow$ $HN(E)$ for $E \in \mathrm{Coh}^{\omega,B}(X)$ is polyhedral on the left, i.e. there are finitely many extremal points $z_0, \ldots, z_m = Z_{\omega,B}(E)$ s.t $HN^{\omega,B}(E)$ lies to the right of the path $z_0 \cdots z_m$.



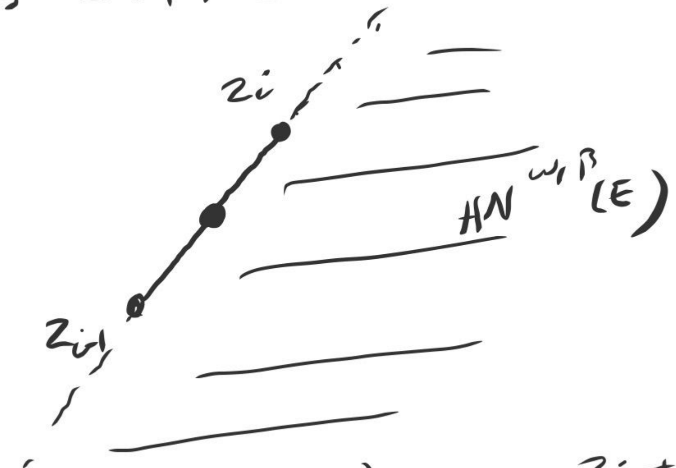
④ **Prop.** the object $E \in \mathrm{Coh}^{\omega,B}(X)$, has a HN filtration w.r.t $Z_{\omega,B}$ iff $HN^{\omega,B}(E)$ is polyhedral on the left.

$\underline{\text{Pf}}$. Assume $HN^{\omega,\beta}(E)$ is polyhedral on the left, for each $i = 1, \dots, m$, we can choose $E_i \subset E$ in $\mathrm{Coh}^{\omega,\beta}(X)$ s.t. $Z_{\omega,\beta}(E_i) = z_i$.

(i) $\underline{E_{i-1} \subset E_i}$. for $\underline{i = 1, \dots, m}$. .

$$E_{i-1} \cap E_i \longrightarrow E_{i-1} \oplus E_i \twoheadrightarrow E_{i-1} + E_i \qquad (*)$$

$Z := Z_{\omega,\beta}$



$$\frac{Z(E_{i-1} \cap E_i) + Z(E_{i-1} + E_i)}{2} = \frac{z_i + z_{i-1}}{2}$$

$E_{i-1} \cap E_i \subset E$, $E_{i-1} + E_i \subset E$ $\rightsquigarrow \begin{cases} Z(E_i \cap E_i) \in HN^{\omega,\beta}(E) \\ Z(E_{i-1} + E_i) \in \end{cases}$

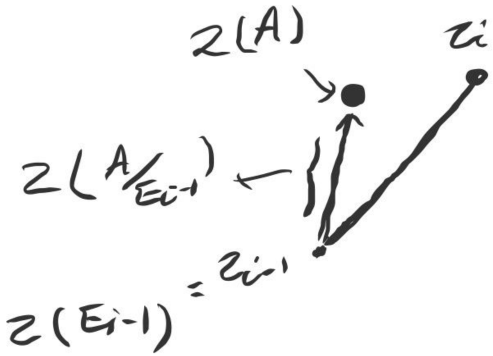
$\underline{Z(E_{i-1} \cap E_i) \in \overline{z_{i-1} z_i}}$

$\underline{E_{i-1} \cap E_i} \subset \underline{E_i} \Rightarrow Z(E_{i-1} \cap E_i) = z_{i-1}$

$\Rightarrow E_{i-1} \cap E_i = E_{i-1}$ $\square$

(ii) E_i/E_{i-1} is semistable w.r.t $Z_{\omega,B}$.

if not, $\exists$ object A inside $E_{i-1} \subset A \subset E_i$ s.t A/E_{i-1} has phase bigger than E_i/E_{i-1}.



but $A \subset E$

$\rightsquigarrow Z_{\omega,B}(A) \in HN^{\omega,B}(E)$ $\lightning$.

* the phase of E_i/E_{i-1} is determined by slope of $\overline{z_{i-1} z_i}$,

so $\phi(E_1/E_0) > \phi(E_2/E_1) > \cdots$

$\Rightarrow$ $0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = E$ is a HN filtration of E

Conversely, assume that we are given a HN filtration of E

$0 = E_0 \subset E_1 \subset \cdots \subseteq E_m = E$, and a subobject $F \hookrightarrow E$,

we only need to show $Z_{\omega,B}(F)$ lies to the right of the path.

$z_0 z_1 \cdots z_m$ where $z_i = Z_{\omega,B}(E_i)$

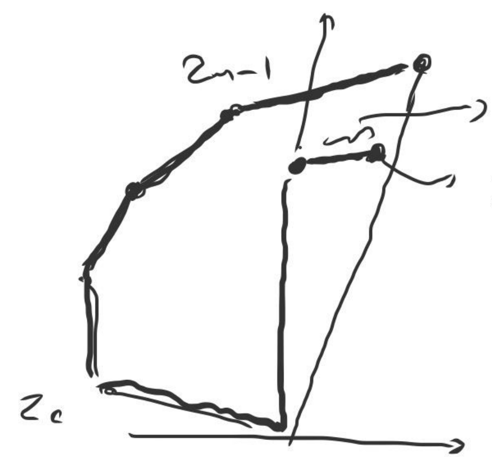
* we prove the claim by induction on the length of HN filtration m.

$F \cap E_{m-1} \subset E_{m-1}$, by induction step, we may assume $Z_{\omega,B}(F \cap E_{m-1})$

lies to the right of the path $z_0 \cdots z_{m-1}$.

Since E/E_{m-1} is semistable. $\Rightarrow Z\left(F/F\cap E_{m-1}\right)$ lies to the right of the

line segment from 0 to $z_m - z_{m-1}$.



$Z(F) = Z(F \cap E_{m-1}) + Z\left(F/F\cap E_{m-1}\right)$

lies to the right of

$z_0 \cdots z_m$.

$\square$

⑤ support property. □

stability conditions on K3 surfaces:

X : algebraic K3 surface / $\mathbb{C}$ $\omega_X \simeq \mathcal{O}_X$, $H^1(X, \mathcal{O}_X) = 0$

$K_{num}(X) = \mathcal{N}(X) = \mathbb{Z} \oplus NS(X) \oplus \mathbb{Z}$.

$E \in D^b(X) \rightsquigarrow$ Mukai vector $v(E) \in \mathcal{N}(X)$

$$v(E) = (ch_0(E), ch_1(E), ch_2(E) + rk(E))$$

The Mukai bilinear form:

$$\langle (r_1, D_1, s_1), (r_2, D_2, s_2) \rangle = D_1 \cdot D_2 - r_1 s_2 - r_2 s_1$$

which makes $\mathcal{N}(X)$ into a lattice of signature $(2, \rho)$

$$\chi(E, F) = \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}(E, F[i]) = -\langle v(E), v(F) \rangle.$$

Lem. If $E \in D^b(X)$ is stable in some stability condition on X, or is $\mu_{\omega,B}$-stable sheaf for some $\omega \in \mathrm{Amp}(X)$, then

$$\dim_{\mathbb{C}} \mathrm{Hom}(E, E[1]) = 2 + v(E)^2 \geq 0$$

with equality precisely when E is spherical, i.e.

$$\mathrm{Hom}(E, E[i]) = \begin{cases} \mathbb{C} & \text{if } i \in \{0,2\} \\ 0 & \text{o/w} \end{cases}$$

Pf. Serre duality,

$$\mathrm{Hom}(E, E[i]) = \mathrm{Hom}(E, E[2-i])^*.$$

Since E is stable, so E lies in the heart of a bounded t-structure.

So $\mathrm{Hom}(E, E[i]) = 0$ for $\underline{i<0}$ and $\underline{i>2}$.

$$\mathrm{Hom}(E, E) \cong \mathbb{C} \overset{SD}{=} \mathrm{Hom}(E, E[2])^*.$$

$$v(E)^2 = -\chi(E,E) = \underline{-2} + \underline{\dim_{\mathbb{C}} \mathrm{Hom}(E, E[1])}$$

$\square$

Rmk. $v(E) = (\mathrm{rk}(E), ch_1(E), ch_2(E) + \mathrm{rk}(E))$ $\Rightarrow \underline{\Delta(E) \geq 2\mathrm{rk}(E)^2 - 2}$

$$v(E)^2 = ch_1(E)^2 - 2\mathrm{rk}(E)(ch_2(E) + \mathrm{rk}(E)) \geq -2$$

$\mathbb{R}$-divisor $B, \omega \in NS(X)_{\mathbb{R}}$ s.t. $\omega \in \mathrm{Amp}(X)$.

$\rightsquigarrow \mathrm{Coh}^{\omega,B}(X) = \langle \mathcal{F}^{\omega,B}[1], \mathcal{T}^{\omega,B} \rangle$.

$\tilde{Z}_{\omega,B} : \mathcal{N}(X) \to \mathbb{C}$

$\tilde{Z}_{\omega,B}(E) = \langle \exp(B + i\omega), v(E) \rangle$.

<u>Ex. 10</u>. Let $\omega = \alpha H$ $\alpha \in \mathbb{R}^{>0}$ if $\underline{\omega^2 = \alpha^2 H^2 > 2}$, then.

$\tilde{Z}_{\omega,B} = Z_{\alpha' H, B}$ where $\alpha' = \sqrt{\alpha^2 - 2/H^2}$.

$\Rightarrow$ If $\underline{\omega^2 > 2}$, then $\tilde{\sigma}_{\omega,B} = (\mathrm{Coh}^{\omega,B}(X), \tilde{Z}_{\omega,B})$ is a stability condition on X.

Prop. The pair $\tilde{\sigma}_{\omega,B}$ is a stability condition on X if for all spherical sheaf E on X with $\mathrm{Im}[\tilde{Z}_{\omega,B}(E)]=0$, one has

$\mathrm{Re}[\tilde{Z}_{\omega,B}(E)] \notin \mathbb{R}_{\leq 0}$

Claim. $\tilde{Z}_{\omega,B}$ is a stability function on $\mathrm{Coh}^{\omega,B}(X)$.

$\mathrm{Im}[\tilde{Z}_{\omega,B}(E)] = \omega \cdot (ch_1(E) - ch_0(E) \cdot B)$.

By def of $\mathrm{Coh}^{\omega,B}(X)$, $\mathrm{Im}[\tilde{Z}_{\omega,B}(E)] \geq 0$. for $E \in \mathrm{Coh}^{\omega,B}(X)$.

if $\mathrm{Im}[\tilde{Z}_{\omega,B}(E)]=0$ $\longrightarrow$ E: torsion sheaf supported in $\dim = 0$. ✓

$\hookrightarrow$ $E = F[1]$ where F is a $\mu_{\omega,B}$-semistable sheaf with $\mu_{\omega,B}=0$

Let $v(F) = (r, c, s)$

$R := \mathrm{Re}[\tilde{Z}_{\omega,B}(F[1])] = -\frac{1}{2r}\left((c^2 - 2rs) + r^2\omega^2 - (c-rB)^2 \right)$

by lemma. $v(F)^2 = c^2 - 2rs \geq -2$

The case F is $\mu_{\omega,B}$-stable. $\rightsquigarrow$

since $\mu_{\omega,B}(F)=0$ we have $\omega \cdot (c - rB) = 0$ $\xrightarrow[\text{Thm}]{\text{Hodge index}}$ $(c-rB)^2 \leq 0$

If F is not spherical, $v(F)^2 \geq 0$. $\Rightarrow$ $R \leq \frac{-1}{2r}(r^2\omega^2) < 0$