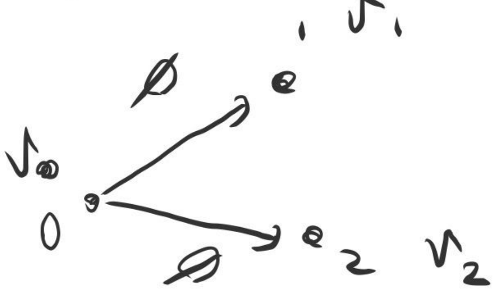
stability for quiver representations:

a finite quiver $Q = (Q_0, Q_1)$ is a directed graph with vertices $Q_0 = \{0, \dots, n\}$ and arrows Q_1.

A quiver representation $\underline{V} = (\underline{V_0}, \underline{V_1})$ is an assignment of k-vector spaces in a set $\underline{V_0}$ to each vertex, and linear maps in a set $\underline{V_1}$ to each edge.

Rep_Q = the category of its representations which is abelian.



we assume Q has no loops or oriented cycles.

$\rightsquigarrow \underline{K(Rep_Q)} = \langle S_i \rangle_{0 \le i \le n}$

where S_i = having one copy of k assigned to the i-th vertex and having 0 assigned to every other vertex

Ex Kronecker quiver P_2 :

$$\begin{tikzcd} \underset{0}{\bullet} \arrow[r, bend left, "x_0"] \arrow[r, bend right, "y_0"'] & \underset{1}{\bullet} \end{tikzcd}$$

$$\begin{tikzcd} V_0 \arrow[r, bend left, "\phi_0"] \arrow[r, bend right, "\phi_1"'] & V_1 \end{tikzcd} \qquad S_0 : \begin{tikzcd} \mathbb{C} \arrow[r, bend left] \arrow[r, bend right] & 0 \end{tikzcd} \qquad S_1 : \begin{tikzcd} 0 \arrow[r, bend left] \arrow[r, bend right] & \mathbb{C} \end{tikzcd}$$

. Pick complex numbers $z_0, \dots, z_n \in \mathbb{H} = \{ R_{>0} \exp(i\pi\phi) \mid 0 < \phi \leq 1 \}$

stability function $Z : K(\mathrm{Rep}\, Q) \longrightarrow \mathbb{C}$

$$Z(\underline{V}) = \sum_{i=0}^{n} \dim V_i \cdot z_i$$

P_2 : $z_0, z_1 \in \mathbb{H}$

(1) $\phi(z_1) > \phi(z_0)$ $\rightsquigarrow$ Q. what are stable objects?

$$\begin{tikzcd} 0 \arrow[r, bend left] \arrow[r, bend right] \arrow[d, hook] & V_1 \arrow[d] \\ V_0 \arrow[r, bend left, "\phi_1"] \arrow[r, bend right, "\phi_2"'] & V_1 \end{tikzcd} \twoheadrightarrow \begin{tikzcd} V_0 \arrow[r, bend left] \arrow[r, bend right] & 0 \end{tikzcd}$$

$\Rightarrow$ the only stable objects are S_0 and S_1

(2) If $\underline{\phi(Z_0) > \phi(Z_1)}$.

• $\underline{\text{claim}}$. the moduli space of stable representation of dim vector $(1,1)$ is isomorphic to $\mathbb{P}^1$.

A representation of class $(1,1)$ is stable iff S_0 is not a subrepresentation

this is equivalent to saying $(\phi_1, \phi_2) \neq (0,0)$.

• for any $\lambda \in \mathbb{C}^*$, $\mathbb{C} \overset{\phi_1}{\underset{\phi_2}{\rightrightarrows}} \mathbb{C}$ and $\mathbb{C} \overset{\lambda\phi_1}{\underset{\lambda\phi_2}{\rightrightarrows}} \mathbb{C}$ are isomorphic

$\Rightarrow$ isomorphism classes of stable representation of class $(1,1)$ parametrized by

$$\{(\phi_1, \phi_2) \in \mathbb{C}^2 \setminus \{(0,0)\}\} / \mathbb{C}^* \cong \mathbb{P}^1$$

$(n+1)$ arrows.

$\underline{P_{n+1}}$: $\underset{v_0}{\bullet} \overset{\vdots}{\rightrightarrows} \underset{v_1}{\bullet}$

If $\phi(Z_0) > \phi(Z_1)$;

the isomorphism classes of representation with dim vector $(1,1)$ is parametrized by $\mathbb{P}^n$.

Example of torsion pair :

$\mathcal{T}$ = representation concentrated at vertex. $S_n^{\oplus k}$

$\mathcal{F}$ = subcategory of representation $\underline{V}$ with $\mathrm{Hom}(S_n, \underline{V}) = 0$, i.e.

the intersection of the kernels of all maps $\phi_j : V_n \to V_i$

going out of V_n is trivial.

$\rightsquigarrow$ $(\mathcal{T}, \mathcal{F})$ is a torsion pair on Rep_Q.

$\rightsquigarrow$ $A^\# = \langle \mathcal{F}[1], \mathcal{T} \rangle$ is heart.

claim. Fix complex numbers $z_0, \dots, z_{n-1} \in \mathcal{H} = \mathbb{R} \times \mathbb{R}^{>0}$ real case

$z_n = -x + i\varepsilon$ for $x, \varepsilon > 0$, then $Z : K(\mathrm{Rep}_Q) \to \mathbb{C}$ is

$\underline{V} \to \sum_{i=0}^{n-1} \dim V_i \cdot z_i + z_n \cdot \dim V_n$

a stability function on $A^\#[-1]$ if $0 < \varepsilon << 1$.

$\langle \mathcal{F}, \mathcal{T}[-1] \rangle$

$Z(S_n[-1]) \in \mathbb{H}$ ✓

$Z(\underline{V}) \in \mathbb{H}$ for $\underline{V} \in \underline{\mathcal{F}}$.

$$\bigcap_j \ker(\phi_j : V_n \to V_i) = \{0\} \implies \sum_{i \neq n} \dim V_i > k \dim V_n$$

for some $\underline{\underline{k > 0}}$

$$\mathrm{Im}[Z(\underline{V})] = \underline{\sum_{i \neq n} \mathrm{Im}[z_i] \cdot \dim V_i - \varepsilon \dim V_n} > 0$$

for $\varepsilon \ll 1$.

Ex. Let C be a smooth projective curve, any object in $D^b(C)$ is the direct sum of its cohomology sheaves.

Pf. We proceed by induction over the length of the complex.

$E \in D^b(C)$, $H^i(E) = 0$ for $i < i_0$.

exact triangle in $D^b(C)$.

$$H^{i_0}(E)[-i_0] \longrightarrow E \longrightarrow E' \longrightarrow H^{i_0}(E)[-i_0+1]. \quad (*)$$

claim. this triangle splits.

$$\mathrm{Hom}_{D^b(C)}(E', H^{i_0}(E)[-i_0]) = \mathrm{Hom}_{D^b(C)}\Big(\bigoplus_{i>i_0} H^i(E')[-i], H^{i_0}(E)[1-i_0]\Big)$$

$$= \bigoplus_{i>i_0} \mathrm{Ext}^{1-i_0+i}(H^i(E'), H^{i_0}(E)) = 0$$

$\square$

X : smooth projective surface. $\omega \in \mathrm{Amp}(X)$, $B \in NS(X)_{\mathbb{R}}$

$\rightsquigarrow \sigma_{\omega,B} = (\mathrm{Coh}^{\omega,B}(X), Z_{\omega,B})$ $\qquad Z_{\omega,B}(-) = -ch_2^B + \frac{\omega^2}{2} ch_0^B + i(\omega \cdot ch_1^B)$.

$$\mathrm{Coh}^{\omega,B}(X) = \langle \underline{\mathcal{F}}^{\omega,B}[1], \underline{\mathcal{T}_{\omega,B}} \rangle$$

<u>Ex.</u> 1. $\mathcal{O}_X$ is minimal in $\underline{\mathrm{Coh}^{\omega,B}(X)}$.

<u>Pf</u>. $\underbrace{F_1 \hookrightarrow \mathcal{O}_X \twoheadrightarrow F_2}$ $\Rightarrow$ $\underline{\mathrm{Im}[Z_{\omega,B}(F_1)] = 0}$

$$0 \to H^{-1}(\cancel{F_1}) \to 0 \to \underline{H^{-1}(F_2)} \to \underbrace{H^0(F_1) \to \mathcal{O}_X \to H^0(F_2)} \to 0$$

$H^0(F_1) = F_1$

$\hookrightarrow$ F_1 is a sheaf. $\Rightarrow$ F_1 is a torsion sheaf supported in dim zero.

torsion-free $\quad H^{-1}(F_2) = 0$

$F_1 = H^0(F_1) = 0$ or $F_2 = H^0(F_2) = 0$ $\lightning$.

Ex. 2. Let E be a $\mu_{\omega,B}$-stable vector bundle on X with $\mu_{\omega,B}(E) = 0$, then $E[1]$ is minimal in $\mathrm{Coh}^{\omega,B}(X)$.

$F_1 \hookrightarrow E[1] \twoheadrightarrow F_2$. torsion-sheaf supp in dim zero.

$$0 \to \cancel{H^{-1}(F_1)} \to E \xrightarrow{d} H^{-1}(F_2) \to H^0(F_1) \to 0 \to \cancel{H^0(F_2)} \to 0$$

$F_2 = H^{-1}(F_2)[1]$, since $F_2 \in \mathrm{Coh}^{\omega,B}(X)$ and $\mu_{\omega,B}(F_2) = 0$

$\Rightarrow$ F_2 is a $\mu_{\omega,B}$-semistable torsion-free sheaf.

$$0 \to H^{-1}(F_1) \to E \twoheadrightarrow \mathrm{Im}\, d$$

$\mu_{\omega,B}$-semistable $\mu_{\omega,B}(\mathrm{Im}\, d) = 0$

$\Rightarrow$ $E = \mathrm{Im}\, d$. $\Rightarrow$ $H^{-1}(F_1) = 0$

$$0 \to E \to \underbrace{H^{-1}(F_2)}_{\text{torsion-free}} \to H^0(F_2) \to 0 \quad \rightsquigarrow \quad \mathrm{Ext}^1(H^0(F_1), \underbrace{E}_{\text{vector bundle}}) \neq 0 \quad \lightning.$$

assum.

$\omega = \alpha H$ where $H \in NS(X)$ is an integral ample divisor.

$\alpha \in \mathbb{R}_{>0}$

$\sigma_{\omega = \alpha H, B}$ when $\alpha \to \infty$.

Lem. Let $E \in \mathrm{Coh}^{\omega, B}(X)$ be $\sigma_{\alpha H, B}$ - semistable for $\alpha \gg 0$, then.

(1) $H^{-1}(E) = 0$ and $H^0(E)$ is a $\mu_{\omega, B}$ - semistable torsion-free sheaf.

(2) $H^{-1}(E) = 0$ and $H^0(E)$ is a torsion-sheaf.

(3) $H^{-1}(E)$ is a $\mu_{\omega, B}$ - semistable torsion-free sheaf and $H^0(E)$ is either zero or a torsion sheaf supported in dim zero.

Pf. Ordering of stability filtration. is the same as the ordering of

$$-\frac{\mathrm{Re}.[Z(-)]}{\mathrm{Im}[Z(-)]}.$$

$$\lim_{\alpha \to \infty} \frac{-\mathrm{Re}[Z_{\alpha H, B}(-)]}{\mathrm{Im}[Z_{\alpha H, B}(-)]} \underset{\text{if } ch_0 \neq 0}{=} \frac{-\frac{\alpha^2}{2} H^2 ch_0(-)}{\alpha H \cdot ch_1^B(-)} = \gamma \qquad (*)$$

- $H^{-1}(E)[1] \hookrightarrow E \to H^0(E)$ in $\mathrm{Coh}^{\omega, B}(X)$.

$*$ If $H^{-1}(E) \neq 0$ and $ch_0(H^0(E)) \neq 0$, then E is $\sigma_{\alpha H, B}$-unstable for $\alpha \to \infty$.

$$\underset{+\infty}{\gamma(H^{-1}(E)[1])} > 0 > \underset{-\infty}{\gamma(H^0(E))}$$

- If $H^{-1}(E) = 0$,

(i) if $ch_0(E) \neq 0$, then E is a torsion-free sheaf. since E is $\sigma_{\alpha H, B}$ semistable.

$\Rightarrow E$ is $\mu_{\omega, B}$-semistable.

$$\left[\underset{\text{has the same ordering}}{\gamma \sim} -\frac{ch_0(-)}{H \cdot ch_1^B} \sim \frac{H \cdot ch_1^B}{ch_0} = \mu_{\omega, B} \right)$$