Lecture 1. - toric geometry crash course.

How to build a toric variety:

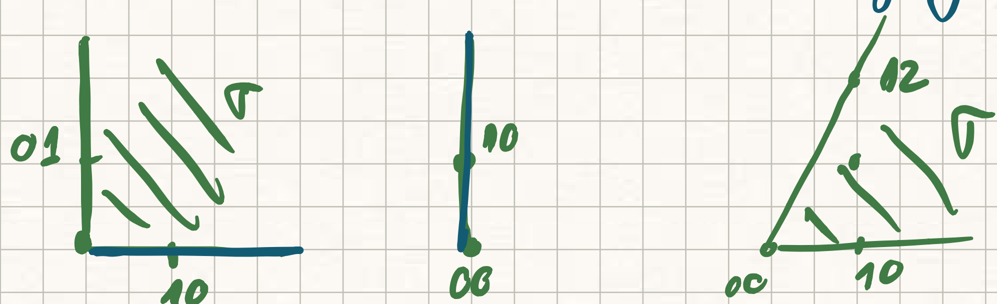
Idea: construct an affine (later projective) variety from a combinatorial object.

OBJ: The cone:

Let $\{v_i\}_{i=1}^{m} \in \mathbb{R}^n$. The set $\sigma = \{ x \in \mathbb{R}^n \mid x = \lambda_1 v_1 + \dots + \lambda_m v_m,\ \lambda_i \in \mathbb{R}_{\geq 0} \}$ is a polyhedral cone. (it's rational if $v_i \in \mathbb{Q}^n$)

- the generators of σ
- $N \simeq \mathbb{Z}^n \subset \mathbb{R}^n$ the underlying lattice.

Ex:



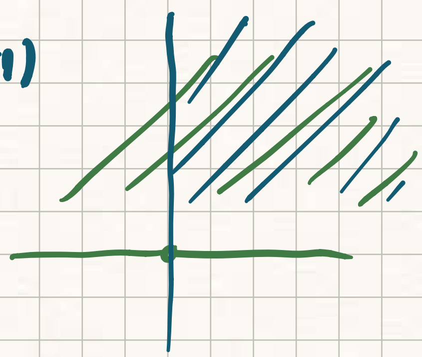
(1) (2) (3)

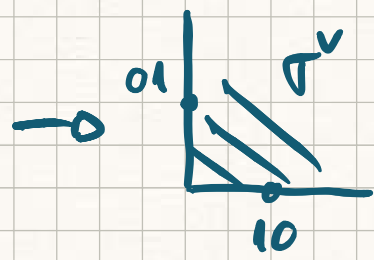
Cone duality: Let $(\mathbb{R}^n)^*$ be the dual of $\mathbb{R}^n$ & $\langle , \rangle$ the usual pairing.

$$\check{\sigma} = \{ u \in (\mathbb{R}^n)^* \mid \langle u, v \rangle \geq 0 \ \forall v \in \sigma \}$$

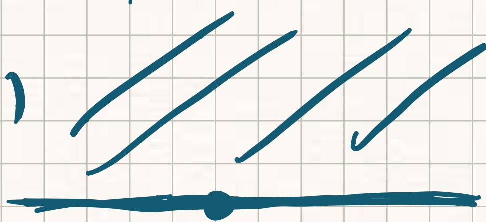
its underlying lattice is $M = \mathrm{Hom}(N, \mathbb{Z}) \simeq \mathbb{Z}^n$.

Their duals:

(1) 

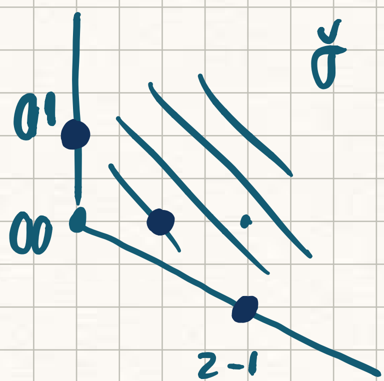


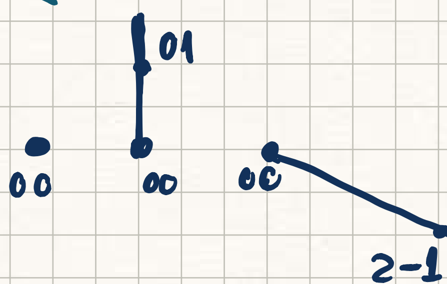
also called lattice of characters.

(2) 

- A cone is **strongly convex** if it doesn't contain a line through the origin.

(Note that the dual of SC cone is not nec. SC)

(3) 

- **Faces** of $\sigma^\vee$: 

& $\sigma^\vee$ itself.

Rk: If σ is a rat'l polyhedral cone $\Rightarrow$ $\sigma \cap N$ has a monoid structure. (f.g.)

$\hookrightarrow$ S a f.g. monoid if $\exists\ a_1, \dots, a_n \in S$ st.

$$\forall s \in S \qquad s = \lambda_1 a_1 + \dots + \lambda_m a_m, \ \lambda_i \geq 0.$$

$\check{\sigma}$ is gen by: $(2,-1), (1,0), (1,0).$

We choose $S_\sigma := \check{\sigma} \cap M.$

→ in general: this choice of generators corresp. to a presentation of the fg. algebra

$$R_\sigma = \mathbb{C}[x_1, \dots x_m] / I_\sigma$$

↳ $X_\sigma = \operatorname{Spec} R_\sigma.$

Rk: The gp. homomorphism $\mathbb{Z}^n \to \overbrace{\mathbb{C}[z_1 \dots z_n, z_1^{-1} \dots z_n^{-1}]}^{\mathbb{C}[z, z^{-1}]}$

$$a = (\alpha_1 \dots \alpha_n) \mapsto \underline{\underline{z_1^{\alpha_1} \cdots z_n^{\alpha_n} =: z^a}}$$

is an isomorphism between $\mathbb{Z}^n$ and <u>monic</u> laurent monomials

→ take $R_\sigma = \{ f \in \mathbb{C}[z, z^{-1}] \mid \operatorname{supp} f \subset \check{\sigma} \cap M \}.$

The trick is to rewrite this as an algebra.

↳ In EX3: $\begin{cases} a_1 = (0\,1) \\ a_2 = (1\,0) \\ a_3 = (2,-1) \end{cases}$ → $R_\sigma = \mathbb{C}[\overbrace{z_2}^{u}, \overbrace{z_1}^{v}, \overbrace{z_1^2/z_2}^{w}]$

$$= \mathbb{C}[u, v, w] /$$

relation: $a_1 + a_3 = 2a_2$ $\qquad (uw - v^2)$.

X_σ = the affine toric variety associated to σ.
$\rightarrow$ if σ is of maximal dim inside $\mathbb{R}^n$,
then $\dim X_\sigma = n$.

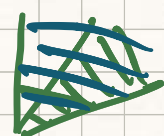
Where is the torus that acts on X_σ?
Let $a_1, \dots, a_m$ be gen of S_σ, $\dim \sigma = n$.
One can embed the alg. torus $\mathbb{T} = (\mathbb{C}^\times)^n$ in X_σ
by sending:

$$t \longmapsto \underbrace{(t^{a_1}, \dots, t^{a_m})}_{} \in (\mathbb{C}^\times)^m$$

$\hookrightarrow$ verifies the equations in I_σ
$\in X_\sigma$.

This induces
an action of $\mathbb{T} \circlearrowright (\mathbb{C}^\times)^m \cap X_\sigma$.

OBJECT #2: FANS



$\hookrightarrow$ A fan $\Sigma \subset \mathbb{R}^n$ is a finite collection of strongly convex polyhedral cones s.t.

- every face of a cone $\sigma \in \Sigma$ is a cone in Σ.
- if σ & σ' are cones $\Rightarrow$ $\sigma \cap \sigma'$ is a common

face of σ & σ'

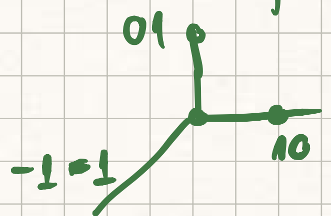
One can build a variety X_Σ by glueing together affine pieces X_σ $\forall\, \sigma \in \Sigma$ if we keep in mind

$$\tau \preccurlyeq \sigma \to \check{\tau} \supsetneq \check{\sigma} \Leftrightarrow S_\tau \supseteq S_\sigma \text{ so } X_\tau \subseteq X_\sigma .$$

"is a face of"

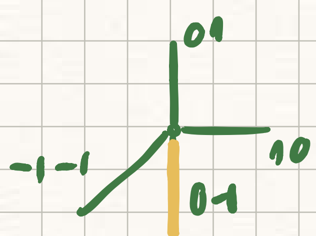
- this also a toric variety.

Examples: (1) Build $\mathbb{P}^2$ from

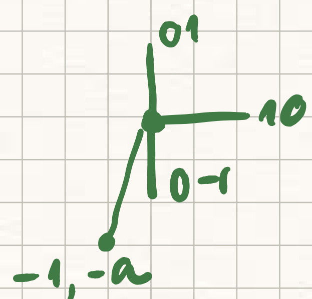


- 3 max cones
- $X_{\sigma_i} \simeq \mathbb{C}^2$
- When glueing them, the familiar charts on $\mathbb{P}^2$ appear.

(2) Build $Bl_{pt}\mathbb{P}^2$ from



(3) Build the Hirzebruch surface $\mathbb{F}_a$ from



Geometrically: Σ complete (cover $\mathbb{R}^n$) $\Longleftrightarrow$ X_Σ is compact

+ dim 2 $\Longleftrightarrow$ X_Σ is projective.

The orbit-cone correspondence:

$$\{\sigma \in \Sigma\} \xleftrightarrow{1:1} \{T_N\text{-orbits}\}.$$

- $$\sigma \longmapsto \mathcal{O}(\sigma) = \mathrm{Hom}_{\mathbb{Z}}(\underbrace{\sigma^\perp \cap M}_{\text{is a sublattice of } M}, \mathbb{C}^*)$$

this is a torus.

- dim $\mathcal{O}(\sigma) = n - \dim(\sigma)$

→ if σ is a ray, its orbit is an $(n-1)$-dim'l torus.

- $$\overline{\mathcal{O}(\tau)} = \bigcup_{\tau \preceq \sigma} \mathcal{O}(\sigma)$$

→ $\overline{\mathcal{O}(\sigma)}$ is a toric var of dim $n-1$.

⟹ it is a <u>divisor</u> on X_Σ.

For a ray ρ_i of Σ, $D_i := \overline{\mathcal{O}(\rho_i)}$ a Weil divisor.

<u>Prop</u>: The D_i generate $\mathrm{Div}_{T_N}(X_\Sigma)$

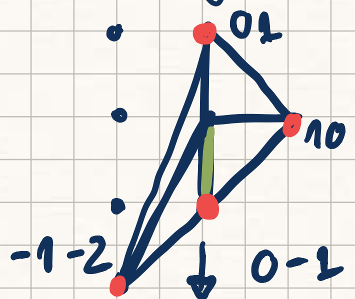
→ in particular:

$$-K_{X_\Sigma} = \sum_{\rho_i \text{ ray}} D_i$$ is a torus-inv. divisor.

Fact: Let $r_i \in \mathbb{Z}^n$ be primitive generators of the rays of X_Σ. If they are the vertices of a convex polyhedron P (which automatically contains 0 in its strict interior), then X_Σ is a Fano variety (possibly singular)

We call a P — cont 0 in its int.
— having primitive vertices

a Fano polytope.



← this is NOT Fano.

Advantages of the fan perspective:

- Singularities of X_Σ are easier to determinate.
  - → isolated.
  - → terminal /canonical, discrepancies
  - → MMP
  - → X_Σ is Gorenstein /$\mathbb{Q}$-Gorenstein.

$\longrightarrow$ a $\overset{max}{\text{cone}}$ $\Leftrightarrow$ a point is singular
if its generators do not form a basis
of $\mathbb{Z}^n$. $\rightarrow$

$\rightarrow$ if $\overset{\forall}{\sigma}$ is simplicial $\rightarrow$ X_Σ is an orbifold
$\nwarrow \Sigma$ $\hookrightarrow$ if σ is singular, its sing. are
cyclic quotient & their order is
$[N : \langle \mathbb{Z}v_1 \oplus \dots \oplus \mathbb{Z}v_m]$.