$\mathbb{C}^{n+1} \setminus Z / (\mathbb{C}^*)^r$

Example:

x_0	x_1	x_2	x_3
1	1	2	0
0	1	3	1

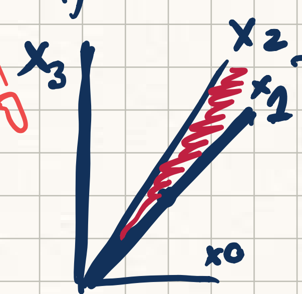
$\mathbb{C}^* \times \mathbb{C}^*$

$\to (\alpha,\beta)(x_0 \dots x_3) = (\alpha x_0, \alpha\beta x_1, \beta^3\alpha^2 x_2, \beta x_3)$

$-K = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \qquad H^0(-K) = \{ x_0^4 x_3^5 , x_0 x_1 x_2 x_3 , x_1^4 x_3 \dots \}$

→ each column is a vector in $\mathbb{Z}^r$ and they span a strictly convex cone there if the toric variety will come from a complete fan.

→ I would like to build a toric Fano variety from this matrix → I need $-K$ to be in the ample cone of X



→ $-K$ is here, so this will be the nef cone.

Then $I = (x_{i_1} \dots x_{i_s} \mid w \in \langle x_{i_1} \dots x_{i_s} \rangle) \subset \mathbb{C}[x_0 \dots x_n]$.

→ $\mathbb{C}^{n+1} \setminus V(I) / (\mathbb{C}^*)^r$.

→ in ex, $\quad I = (x_1 x_2, x_1 x_3, x_0 x_2, x_0 x_3)$

- $\dim X = \#\text{columns} \setminus \#\text{rows}$

→ X here will be a surface.

Charts: What are the open sets that cover

$\mathbb{C}^{n+1} \setminus V(I)$?

$x_1 x_2 = 0$
$x_1 x_3 = 0$
$x_0 x_2 = 0$
$x_0 x_3 = 0$

$\Leftarrow$ Want to remove this

$\{x_1 \neq 0, x_2 \neq 0\} =: U_{12}$ avoids $V(I)$, therefore is a chart on X.
The others are U_{13}, U_{02}, U_{03}.

How to obtain this kind of presentation from a fan?

Suppose Σ generated by $u_1 \dots u_r$. Suppose the $\{u_i\}$ generate the lattice N as a group.

Then we have an exact sequence:

$$0 \to L \to \mathbb{Z}^r \xrightarrow{f} N \to 0, \qquad N \simeq \mathbb{Z}^m$$

and its dual:

$$0 \leftarrow L^\vee \xleftarrow{D} (\mathbb{Z}^r)^* \leftarrow M \leftarrow 0, \qquad M \simeq (\mathbb{Z}^m)^*$$

$L^\vee = Cl_{T_N}(X_\Sigma) \hookleftarrow \mathbb{Z}^k$, $(\mathbb{Z}^r)^* = Div_{T_N}(X_\Sigma)$

looks familiar! This is $(*)$ from lecture 2.

D = the divisor homomph of X_Σ.

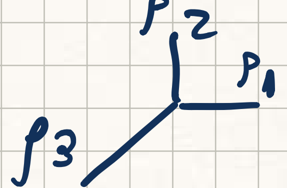
D is dual to a group homomph

$$\underbrace{(\mathbb{C}^*)^k}_{\text{torus w/ character group } \mathbb{L}^\vee} \longrightarrow (\mathbb{C}^*)^r$$ which extends to $\mathbb{C}^r$.

$\to$ We obtain that $X_\Sigma = \mathbb{C}^r /\!/ (\mathbb{C}^*)^k$.

If I want X_Σ to be Fano, I make sure that I choose the nef/ample cone to be where $-K$ is.

Easy ex : $\mathbb{P}^2$.



$$\mathbb{Z}^3 \xrightarrow{f} \mathbb{Z}^2 \longrightarrow 0$$
$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

$\hookrightarrow$ dualise

$$0 \leftarrow \mathbb{L}^\vee \xleftarrow[D]{?} \mathbb{Z}^3 \xleftarrow{\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ -1 & -1 \end{pmatrix}} \mathbb{Z}^2 \longleftarrow 0$$

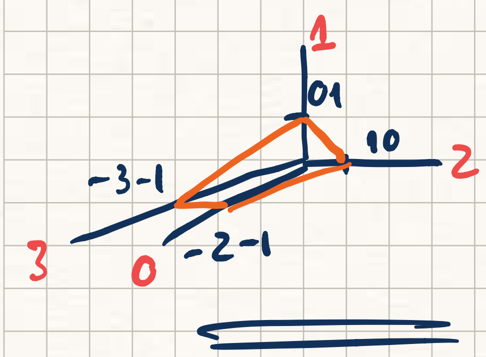
$\hookrightarrow$ this is $\mathbb{Z}$

So ~~B~~ D is a 3×1 matrix. $D = (a_0, a_1, a_2)$

I just want ~~Im~~ Im to be the ker D.

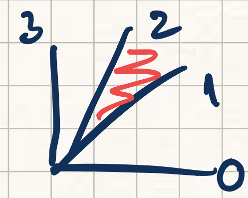
$$(a_0\ a_1\ a_2) \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ -1 & -1 \end{pmatrix} = 0 \to (a_0\ a_1\ a_2) = \underline{(1\,1\,1)}.$$

Ex2: (homework)



I obtain $\longrightarrow$

x_0	x_1	x_2	x_3
1	1	2	0
0	1	3	1



} secondary fan.

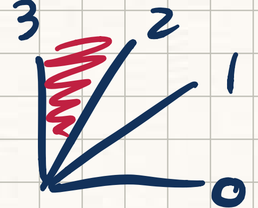
($\rightarrow$ if I choose a different chamber then I will obtain something birational to this.

When I choose a stability condition ω, to build the fan Σ:

$$\sigma \in \Sigma \iff \omega \in \langle D_i \mid i \notin \sigma \rangle.$$

$\rightarrow$ What happens if I choose $\omega \in (2,3)$?

Then $I = (x_{i_1} \dots x_{i_s} \mid \underline{\omega} \in \underline{\langle x_{i_1} \dots x_{i_s} \rangle}) \subset \mathbb{C}[x_0 \dots x_n]$.



Which cones are in this fan Σ'?

$$I = (x_2 x_3,\ x_1 x_3,\ x_0 x_3)$$

$$V(I) = \{x_3 = 0\} \cup \{x_2 = x_1 = x_0 = 0\}.$$

$\rightarrow$ if I want to remove $V(I)$, I have to set $x_3 = 1$.

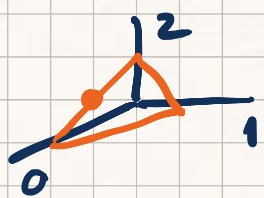
$$\begin{array}{cccc} x_0 & x_1 & x_2 & x_3 \\ \hline 1 & 1 & 2 & 0 \\ 0 & 1 & 3 & 1 \end{array} \quad \to \quad \mathbb{C}^4 \setminus V(*) / (\mathbb{C}^*)^2$$

$$[x_0 : x_2 : x_3 : 1]$$

$\to X_{\Sigma'} = \mathbb{C}^3 \setminus \{0\} / \mathbb{C}^*$

with weights $(1,1,2)$

$\Rightarrow$ I've obtained $\mathbb{P}(112)$



which is exactly the fan of $\mathbb{P}(112)$.

$\to X_\Sigma$ is a blow-up of $X_{\Sigma'}$ at a point.

Complete intersections in toric varieties.

$\to$ Want to build Fanos $X \subset F$ as a complete intersection.

(F: a toric variety)

- take $X \subset F$ a C.I of general elements in $|L_1| \ldots |L_c|$ where $L_i \in \mathbb{L}^\vee$ are line bdles on F.
- then the space of sections $H^0(F, L_i)$ is the subspace of $\mathbb{C}[x_1 \ldots x_n]$ w/ basis consisting of monomials $x^v \in \mathbb{C}[x_1 \ldots x_r]$

where $v \in (\mathbb{Z}^*)^k$ has homogeneity type L_i.

Let $f_i \in H^0(F, L_i)$, then $V(f_1 \ldots f_c)$ is stable under $(\mathbb{C}^*)^k$-action & we consider the subvar $X = \left(V(f_1 - f_c) \setminus \underline{V(I)}\right) / (\mathbb{C}^*)^k \subset F$.

- quasismooth $\to$ $V(f_1 - f_c) \setminus V(I) \subset \mathbb{C}^n \setminus V(I)$ is smooth of codim c.
- well-formed.

Ex:

x_0	x_1	x_2	x_3
1	1	2	0
0	1	3	1

$L_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad L_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

X is given by

$$\begin{cases} L_1: & \bullet x_2 + \bullet x_0 x_3 = 0 \\ L_2: & x_0 = 0. \end{cases}$$

$L_1 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

$\to$ no toric varieties for a bit.

Fano varieties

$\to$ have been classified in the smooth case in dim ≤ 3.

$\to$ Focus on $\mathbb{Q}$-Fano 3folds: $\mathbb{Q}$-Gorenstein w/ @ most terminal sings.

we know these are cyclic quotient singularities $\mathbb{C}^3/\mu_r$ w/ weights $(1, a, -a)$

$\to \frac{1}{r}(1, a, -a)$.

(Altınok, Brown, Iano-Fletcher, Reid, Suzuki)

$\hookrightarrow$ classify all of their Hilbert series. $H^0(X, -mK_X)$

There are 55,000 of these on Graded Ring Database.

– we don't know how many exist.

How would we find such a variety.

- Find m such that
  - $-mK_X$ is v. ample & embed X into $\mathbb{P}^N$.

  Notice you may get a v. high embedding codim.

- We use the Graded ring str.

  $\to$ embed into wps $\underline{\underline{\mathbb{P}(a_0 \ldots a_n)}}$.

  still too high.

  $\Downarrow$

  after codim 5, unless $-K$ is further divisible in $Cl(X)$ can't construct these.

$X \subset \mathbb{P}^n$ is codim 2 $\Rightarrow$ X is CI. (Serre)

X 3 $\to$ it is of Pfaffian format.