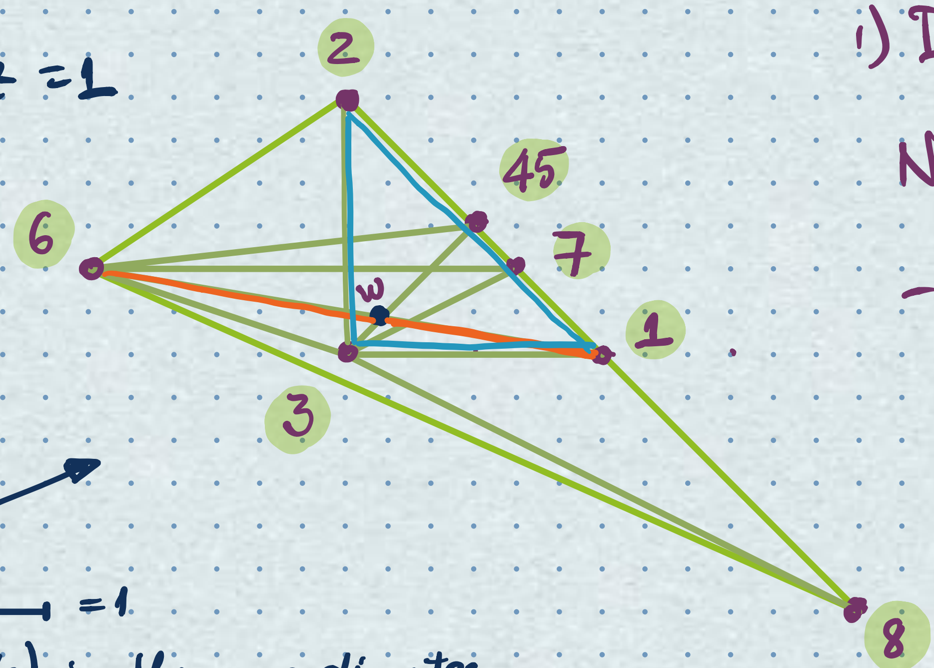
1) Study the singularities of the following toric variety Y:

$$\begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ \hline 1 & 0 & 0 & 1 & 1 & -3 & 2 & 2 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{array} \quad , \quad \underline{w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}.$$

$\dim Y = 5$

$D_i : \{x_i = 0\}$

$x + y + 5z = 1$



1) Is Y Fano?

$\mathrm{Nef}\, Y = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$-K = \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}$

$\vdash\!\!\dashv = 1$

$w = (1/7, 1/7)$ in these coordinates.

2) Study the singularities of the complete intersection of $L_1 = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$ and $L_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ inside Y.

Recall from weighted projective space charts:

$y \neq 0 \neq y = 1$

$\mathbb{P}(1\,1\,3)$ with coordinates x_0, x_1, y

$\hookrightarrow (\alpha x_0, \alpha x_1, \alpha^3 y) \sim (x_0 : x_1 : y)$

set $y = 1$: then the point

$$(x_0 : x_1 : 1) \sim (\mu x_0 : \mu x_1 : \underbrace{\mu^3 \cdot 1}_{1}) \sim$$

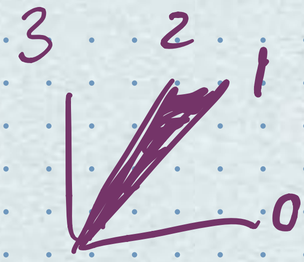
$$\sim (\mu^2 x_0 : \mu^2 x_1 : \underbrace{\mu^6 \cdot 1}_{1})$$

μ is a 3rd root of 1

So 3 pts instead of 1!

$U_3 = \{y \neq 0\} = \{(x_0 ; x_1 ; 1)\}/\mu_3$ w/weights 1,1.

$\hookrightarrow$ has a $\frac{1}{3}(11)$ singularity at its origin.



$\rightarrow$ in general: to find the action solve $\alpha^3 = 1$

1) what are the charts of Y?

U_{ij}

$\|$

$\{x_i \neq 0$

$U_{16} = \{x_1 \neq 0, x_6 \neq 0\}$

$\{x_j \neq 0\}$

$U_{34} = \{x_3 \neq 0, x_4 \neq 0\}$. 12

U_{35} 02

$U_{123} = \{x_1 \neq 0, x_2 \neq 0, x_3 \neq 0\}$ 31

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is also a chart.

U_{367}, U_{687}, U_{268}, U_{238}, U_{237},

U_{468}, U_{568}.

$\to$ all are singular except for U_{123}.

Look at $\underline{268}$ $\quad x_2 \neq 0$

$$\begin{matrix} 0 & -3 & 2 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{matrix} \qquad \begin{matrix} x_6 \neq 0 \\ x_8 \neq 0 \end{matrix}$$

$$(\alpha, \beta, \gamma) \cdot (x_2, x_6, x_8) =$$

$$= (\beta x_2, \alpha^{-3}\beta\gamma x_6, \alpha^2\beta^{-1} x_8)$$

want $U_{268} = \{x_2 = x_6 = x_8 = 1\}$ / ?

$$\begin{cases} \beta = 1 \\ \alpha^{-3}\beta\gamma = 1 \\ \alpha^2\beta^{-1} = 1 \end{cases} \to \begin{cases} \beta = 1 \\ \gamma = \alpha^3 = \alpha \\ \alpha^2 = 1 \end{cases}$$

$\alpha^2 \beta = 1$

α and γ are both (the same $\sqrt[2]{1}$).

The chart has a

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \end{pmatrix} \simeq \frac{1}{2}(1111) \times \mathbb{C}$$

$$x_1 \quad x_3 \quad x_4 \quad x_5 \quad x_7$$

$\alpha^2 \cdot \beta \,|\, x_7$

→ no more action here.

→ What is happening: x_7 is a curve of singularities of type $\frac{1}{2}(1111)$

- the curve is $\{x_1 = x_3 = x_4 = x_5 = 0\} = C_1$
- we find three other curves:

$C_2 = \{x_1 = x_4 = x_5 = x_6 = 0\} \in U_{238} \cup U_{237}$

$C_3 = \{x_1 = x_2 = x_3 = x_7 = 0\} \in U_{468}, U_{458}$

$C_4 = \{x_1 = x_2 = x_6 = x_7 = x_8 = 0\} \in U_{34} \cup U_{35}$

$\downarrow$

non $\mathbb{Q}$-factorial charts.

On U_{34}:

$$\begin{array}{cc} 3 & 4 \\ \hline 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{array}$$

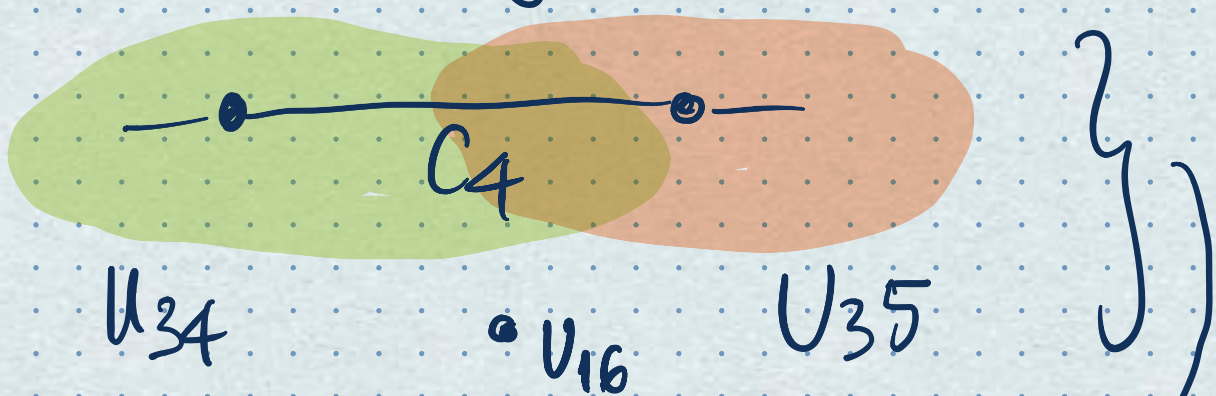
$$(\gamma \cdot x_3, \alpha\beta \cdot x_4)$$

$$\begin{cases} \gamma = 1 \\ \alpha\beta = 1 \to \alpha = \beta^{-1}. \end{cases}$$

$$\mathbb{C}^*_\alpha \circlearrowright \begin{pmatrix} 1 & -1 & 0 & -4 & 1 & 3 \end{pmatrix}$$
$$\quad\ \ x_1 \ \ x_2 \ \ x_5 \ \ x_6 \ \ x_7 \ \ x_8$$

$$\left(\alpha x_1, \frac{1}{\alpha} x_2, x_5, \frac{1}{\alpha^4} x_6, \alpha x_7, \alpha^3 x_8\right)$$

which propagates along C_4



Total: 7 points & 4 curves

all the non
$\mathbb{Q}$-factorial locus is
avoided by a general
section of $L_1 \oplus L_2$.

Call this variety X $\to$ Its singularities will be:

$\frac{1}{5}(1,1,4)$, $\frac{1}{3}(1,1,2)$, $\frac{1}{2}(1,1,1)$.
$\times 2$

Can write:

L_1: $x_4^2 + x_5^2 + x_4x_5 + x_4x_2^3 + \dots = 0$

L_2: $x_3x_4 + x_3x_5 + x_1x_2x_3 + \dots = 0$.

$C_4 = \{x_1 = x_2 = x_6 = x_7 = x_8 = 0\}$

$\in U_{34}, U_{35}$ $\to$ can assume x_3 invertible.

$L_1|_{C_4}$: $x_4^2 + x_5^2 + x_4x_5 = 0$ $\to$ no point.
$L_2|_{C_4}$: $x_4 + x_5 = 0$,

$\hookrightarrow$ assume $\in U_{35} \to x_3 = x_5 = 1$

$\to x_4 = k$

Substitute $(k, 1)$ in the 1st eqn.

$c_1 k^2 + c_2 + c_3 \cdot k = 0.$

for <u>all</u> $c_1, c_2, c_3 \in \mathbb{C}^*$.