Talk 1

§ 0.1 Goals

1- Understand the local and global geometry of moduli spaces of maps to $\mathbb{P}^n$; i.e study deformations, local equations, irreducible components of $\overline{M}_{g,n}(\mathbb{P}^2, d)$.

2- See in the case of genus 1 and 2 stable maps how to exploit ideas and techniques coming from log / tropical geometry to resolve singularities of moduli spaces in "a modular fashion"
↑ i.e the resolution itself is a moduli space

§ 0.2 Why should you care ?

1. Enumerative geometry

Gromov-Witten invariants are difficult to compute in genus $g \geq 1$ and their enumerative meaning is unclear.

This is because $\overline{M}_{g,n}(\mathbb{P}^2, d)$ is very singular, and it has many components parametrizing degenerate objects.

Working on a desingularization of $\overline{M}_{g,n}(\mathbb{P}^2, d)$
- makes computations more accessible
- allow us to define invariants with a better enumerative meaning

2. Log techniques are useful to study compactifications

Motto [From "Handbook of moduli"] : log geometry controls degenerations and helps to "cut down" the number of components of a modular compactification similar techniques to those we'll see are useful to study

- compactifications of moduli of abelian differentials
- moduli of K3 surfaces
- compactified Jacobians and extensions of Abel-Jacobi maps
- birational geometry of moduli of curves . . .

§1. Definitions & First examples

Def : A n-pointed, genus g prestable curve $(C, p_1, .. p_n)$ is a projective, connected, reduced curve, at worst nodal together with n distinct smooth points called markings.

E.g.

$g = 3 = \#$ holes

A family of pre-stable marked curves over S is:

$$\begin{tikzcd} \mathcal{C} \arrow[d, "\pi"] \\ S \arrow[u, bend left, "\sigma_1, .. \sigma_N"] \end{tikzcd}$$

- π is flat
- $\sigma_1, .. \sigma_N$ are smooth section
- Each fiber is a pre-stable curve

E.g.:

$\Sigma = [0:1:0] \; \forall t$

$S = \mathbb{A}^1$ $\quad \{zy^2 + x^3 + x^2 z + t z^3 = 0\} \subseteq \mathbb{P}^2_{[x:y:z]} \times \mathbb{A}^1_t$

Def : Dual graph of a pre-stable curve is

$\Gamma = (V, E, L, g: V \to \mathbb{N})$ where

V = vertices ⟷ correspond to irreducible cpts of C

E = edges ⟷ " " nodes in C

L = legs ⟷ " " markings

$g: V \to \mathbb{N}$ Recall the genus of a given cpt.

Def. A stable mep consist with target X consist of:

(i) e n-marked pre-stable curve $(C, p_1, \dots p_n)$

(ii) A morphism $f: C \to X$ s.t

stability of f and they ensure $|\mathrm{Aut}(f)| < \infty$

- if $D \subseteq C$ is e component of $g=0$, $D \simeq \mathbb{P}^1$ and $f(D) = pt$, i.e D is contracted $\Rightarrow \boxed{\# p_i \in D + \# \text{nodes on } D \geq 3}$ = # of special pts
- if $E \subseteq C$ has $g(E) = 1$ and $f(E) = pt$ $\Rightarrow \boxed{\# p_i \in E + \# \text{nodes} \geq 1}$

A Femiey over S is given by

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\pi"] & X \times S \\ S \arrow[u, bend left, "\sigma_i"] \end{tikzcd}$$

s.t $\forall s$ $C \xrightarrow{f_s} X$ is e stable mep

From now on we will always consider $X = \mathbb{P}^2$

we soy thet f nes degree d if $f_*[C] = d[L]$ <-> generically $F(C) \cap \overset{\text{hyp. in } \mathbb{P}^2}{H} = \overset{d}{\text{pts}}$

Recall thet:

When the target is $\mathbb{P}^2$ we have en equivalence of the Following to sets of doto:

$(C, p_1, \dots p_n)$ pre-stable + $C \xrightarrow{f} \mathbb{P}^2_{[x_0:\dots:x_2]}$ degree d
stable mep

$\updownarrow$

(↓) set $\mathcal{L} = f^*\mathcal{O}(1)$
$s_i = f^* x_i$

(↑) $C \xrightarrow{f} \mathbb{P}^n$
$c \to [ev_c(s_0) : \dots ev_c(s_2)]$

$(C, p_1, \dots p_n)$ pre-stable + $\mathcal{L}$ line bundle of degree d on C
$\boxed{s_0, \dots s_2 \in H^0(C, \mathcal{L})}$ not simuetaneusly vanishing

$\omega_C(p_1 + \dots + p_n) \otimes \mathcal{L}^{\otimes 2}$ ample
stability condition

Exemple 0.1

$C = \mathbb{P}^1_{[t_0:t_1]}$ $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(3)$

$$\boxed{\begin{array}{cccc} t_0^3 & t_0^2 t_1 & t_0 t_1^2 & t_1^3 \\ \shortparallel & \shortparallel & \shortparallel & \shortparallel \\ s_0 & s_1 & s_2 & s_3 \end{array}}$$

The mep $\mathbb{P}^1 \to \mathbb{P}^3$

The map

$p \to [ev_p(s_0) : \dots : ev_p(s_3)]$

is the embedding of $\mathbb{P}^1$ as a twisted cubic $[F] \in \overline{M}_{0,0}(\mathbb{P}^3, 3)$

Example 0.2

$C =$ 

$\boxed{\mathcal{L} = \mathcal{O}_E(3p)}$ Fact : $H^0(E, \mathcal{L})$ has 3 sections $s_0 \; s_1 \; s_2$ not vanishing sim

IF we consider

$E \to \mathbb{P}^2$

$p \to [ev_p(s_0) : ev_p(s_1) : ev_p(s_3)]$

Hartshorne "Alg. geom"

The $F(E)$ is embedded in $\mathbb{P}^2$ as the cubic $\boxed{zy^2 + x^3 + ax^2z + bz^3 = 0}$ and p goes to $[0:0:1]$

a, b depend from the elliptic curve $[F] \in \overline{M}_{1,0}(\mathbb{P}^2, 3)$

$\tilde{C}$

Example 1

$C =$ 

$q = [0:1]$, $\mathcal{L}$ is a line bundle s.t $\mathcal{L}|_E \simeq \mathcal{O}_E$ and $\mathcal{L}|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(3)$

The section $H^0(\mathcal{L})$ are the sections of $H^0(E, \mathcal{O}_E) \oplus H^0(\mathbb{P}^1, \mathcal{O}(3))$ which take the same value at q

Take

$$\boxed{\begin{aligned} s_0 &= (-1, -t_1^3 + t_0^2 t_1) \\ s_1 &= (0, t_0^3 - t_0^2 t_1) \\ s_2 &= (1, t_1^3) \end{aligned}} \Rightarrow \begin{aligned} F : C &\to \mathbb{P}^2 \\ p &\to [ev_p(s_i)] \end{aligned}$$

$F(E) = [-1:0:1]$, $F(\mathbb{P}^1)$ is the curve $\boxed{zy^2 - x^3 - x^2z = 0}$

- E is contracted

i.e 

- the map from $\mathbb{P}^1$ to its image is the normalization $[F] \in \overline{M}_{1,0}(\mathbb{P}^2, 3)$

Example 2

$C =$ 

$\mathcal{L}|_Z = \omega_Z$

$\mathcal{L}|_{\mathbb{P}^1} = \mathcal{O}(2)$

$\deg(\mathcal{L}) = 4$

Fact

Every genus 2 curve admits a map to $\mathbb{P}^1$

$\boxed{Z \xrightarrow[\tau]{2:1} \mathbb{P}^1}$ and $\boxed{\begin{matrix} \tau^* \mathcal{O}(1) \\ \| \\ \omega_Z \end{matrix}}$ is the canonical

hyperelliptic cover

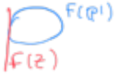
So ω_Z has 2 global sections v_0 and v_1

vanish at q ↓ v_0

↑ v_1 does not vanish at q

choose in $H^0(\mathcal{L})$ $F(C) \subset \mathbb{P}^2$

$$\boxed{\begin{aligned} s_0 &= (u_0, t_0 t_1) \\ s_1 &= (u_1, ev_q(u_1) \cdot t_1^2) \\ s_2 &= (0, t_0^2) \end{aligned}} \Rightarrow \quad F(C) = ..[x:y:z] \quad \boxed{z(zy - \lambda x^2)}$$



$[f] \in \overline{M}_{2,0}(\mathbb{P}^2, 4)$

- Z cover 2:1 the line $z=0$
- $\mathbb{P}^1$ is embedded as a smooth quadric

§ 2. Deformation theory

We have seen some examples of points of

$$\overline{M}_{g,n}(\mathbb{P}^2, d) = \left\{ (C, p_1, \dots p_n) \xrightarrow{F} \mathbb{P}^2 \;\middle|\; \begin{array}{l} F \text{ is stable} \\ \text{of deg } d \end{array} \right\} / \simeq$$

$$\simeq \left\{ (C, p_1, \dots p_n, \mathcal{L}, s_0, \dots s_2) \;\middle|\; \begin{array}{l} s_i \text{ do not venish simult.} \\ \mathcal{L} \text{ has deg} = d \\ \omega_C(p_1 + \dots + p_n) \otimes \mathcal{L}^{\otimes 2} \text{ ample} \end{array} \right\} / \simeq$$

In order to understand the structure of the moduli space we need to understand how the points fit together if families.

We begin by studing deformations, i.e families over fat points $\{ S = \mathrm{Spec}(\mathbb{C}[t]/t^n), s_0 = \mathrm{Spec}(\mathbb{C}) \}$(if you prefer think (S, s_0) germ of cx mfd

§ 2.0 : Generalities

We are mostly interested in

(I) First order deformations, i.e deformations over $D_\varepsilon := \mathrm{Spec}(\mathbb{C}[\varepsilon]/\varepsilon^2)$. ↔

(II) Understand when, given a deformation over D_ε we can extend it to a deformation over a larger base scheme, i.e understand obstructions (e.g over $\mathrm{Spec}(\mathbb{C}[t]/_n \quad n > 2)$)

Indeed:

(I) The set of deformations (up to iso) over D_ε is the tangent space to the moduli space M at the point $\boxed{0 \to M}$

(II) The presence of obstructions tell us that we are at a point of the moduli space which $\boxed{\text{might be singular}}$

$\frac{\mathbb{C}[\varepsilon]}{\varepsilon^3}$

$a, b \in \mathbb{C}$

Cartoon

M looks like

and we look at Def_p

$\frac{\mathbb{C}[x,y]}{xy}$

$a\varepsilon \leftarrow x$

$b\varepsilon \leftarrow y$

$\mathbb{C}[\varepsilon]/\varepsilon^2 \leftarrow \frac{\mathbb{C}[x,y]}{y \cdot x}$

$\Rightarrow$

$$\mathrm{Def}_p(D_\varepsilon) = \underset{0 \to p}{\mathrm{Hom}}(D_\varepsilon, M) \simeq \mathbb{C}^2 \simeq \left(\frac{m}{m^2}\right)^\vee$$

Zariski tangent space

But we can't follow any of the tangent directions any longer: $\boxed{\text{obstructions}}$

In this example the obstructions are contained in the one-dim vector space

$$\boxed{V = \mathrm{Hom}(\mathcal{I}_{M \subset \mathbb{A}}, \mathcal{O}_p)}$$

This mean that $\exists$ a map

$$ob : \boxed{\mathrm{Def}_p(D_\varepsilon) \longrightarrow V} \quad \text{s.t}$$

a deformation ξ lifts $\Leftrightarrow$ $\boxed{ob(\xi) = 0}$

dim $M = 1$ dim obs space = # of eq of M in a smooth amb.

§ 2.1 Deformations of nodal curves

Let C a curve, + markings $(C, p_1, \cdots p_n)$

$$\mathrm{Def}_C(Y, y_0) = \left\{ \begin{array}{c} \mathcal{C} \\ \downarrow \pi \\ Y \end{array} \middle| \begin{array}{c} \pi \text{ flat} \\ \mathcal{C}_{y_0} \simeq C \end{array} \right\} / \simeq$$

$H^1(\mathcal{C}, T_C(-p_1 \cdots -p_n))$

Theorem [ACG II]

- If C is smooth, $\boxed{\mathrm{Def}_C(D_\varepsilon) \simeq H^1(C, T_C)}$

$T_C = \mathcal{H}om(\Omega_C, \mathcal{O}_C)$

idea of proof

Fact: For U an affine smooth scheme, every deformation $\begin{array}{c} \mathcal{U} \\ \downarrow \\ D_\varepsilon \end{array}$ is trivial, i.e $\boxed{\mathcal{U} \xrightarrow{\simeq} U \times D_\varepsilon}$

Let $\{U_i\}$ an affine cover of C and $\boxed{\begin{array}{c}\mathcal{C}\\ \downarrow\\ D_\varepsilon\end{array}}$ a def.

$\Rightarrow$ we have isomorphism

$$\boxed{\mathcal{U}_i \xrightarrow{\varphi_i} U_i \times D_\varepsilon}$$

and thus automorphisms

$$\boxed{\psi : U_{ij} \times D_\varepsilon \longrightarrow U_{ij} \times D_\varepsilon}$$ which

are the identity when restricted to $\varepsilon = 0$

$$\Rightarrow \boxed{\psi(b_0 + \varepsilon b_1) = b_0 + \varepsilon(b_1 + \Theta_{ij}(b_0))}$$

For some addive U_{ij} morphism Θ_{ij}

One can check that $\boxed{\psi(b_0 \cdot b_0') = \psi(b_0)\psi(b_0')}$

$\Rightarrow \Theta_{ij} \in \Gamma(U_{ij}, \mathcal{H}om(\Omega_C, \mathcal{O}_C))$ Θ_{ij} are derivations

and that $\boxed{\Theta_{ij} + \Theta_{jk} + \Theta_{ki} = 0}$

$$\Rightarrow \boxed{\{\Theta_{ij}\} \in H^1(C, T_C)}$$

Reverse eingeneering the argument tell us that given $\Theta \in H^1(C, T_C)$ we get a deformation

gluing the local trivial deformation using $\psi_{ij} = id + \varepsilon(b_1 + \Theta_{ij})$ □

- If C is nodal, $\boxed{Def_C(D_\varepsilon) \simeq Ext^1(\Omega_C, \mathcal{O}_C)}$ and we have a s.e.s

$$0 \to H^1(C, T_C) \to Ext^1(\Omega_C, \mathcal{O}_C) \to H^0(C, \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)) \to 0$$

$\|$ deformations which are locally trivial

$H^1(C, T_C(-p_1, \cdots -p_n))$

$\uparrow$ parametrizes deformations which smoothen the node

$\mathbb{C}^{\boxed{\# \text{ nodes}}}$

$xy - t = 0$

- There are $\boxed{\text{no obstructions}}$ to lift deformations of nodal curves.

- The infinitesimal automorphism, i.e $T_{[C]} \mathrm{Aut}(C)$ is given by $\boxed{H^0(C, T_C)}$ $\Rightarrow H^0(C, T_C(-p_1 \dots -p_n))$)
  $\overset{\shortparallel}{\mathrm{Ext}^0}(\Omega_C, \mathcal{O}_C)$

Notice that $\dim \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)$ can jump as choosing different nodal curves C of some fixed genus but

$$\boxed{\mathrm{Ext}^1(\Omega_C, \mathcal{O}_C) - \mathrm{Ext}^0(\Omega_C, \mathcal{O}_C) := \chi(T_C)}$$

is constant once we fixed the genus (and the # of markings)

↳ moduli stack of pre-stable curves

$$\mathcal{M}_{g,n} = \{(C, p_1 \dots p_n) \text{ pre-stable marked}\}/\simeq$$

Having studied the deformations we now know that the space

→ the tangent space at $[C]$ $T_{[C]}\mathcal{M}_{g,n} = \mathrm{Ext}^1(\Omega_C(\Sigma p_i), \mathcal{O}_C)$)
if the dim of the Automorphism is 0

$\Rightarrow$ $\boxed{\text{This is the dim of } \mathcal{M}_{g,n} \text{ at } [C]}$

→ has dimension $= \chi(T_C(-p_1 \dots -p_n)) = 3g - 3 + n$

→ it is smooth (no obstruction)

→ Since there are automorphism, and possibly infinitely many (e.g $C =$  $\mathrm{Aut}(C) = \mathbb{C}^*$)

$\mathcal{M}_{g,n}$ is an Artin stack

→ If we look around a point $[C]$ we can choose a scheme U together with a smooth map $U \to \mathcal{M}_{g,n}$ || chart

Describing the stack around $[C]$ means describe U.

Lemma : There exist a unique local complete $\mathbb{C}$-algebra which is smooth of dim m

$$R = \mathbb{C}[[t_1, .. t_m]]$$

⇒ Locally analytically the chart U looks like.

$$\boxed{U = \mathbb{A}^{\dim H^1(T_C C_p))} \times \mathbb{A}_{t_1, .. t_k}}$$

called smoothing parameters

where $k = \#$nodes

If $$\begin{tikzcd} \mathcal{C}_{|U} \arrow[d, "\pi"'] \\ U \end{tikzcd}$$ is the universal curve over U (i.e any family of curve is pull-back of π)

and

If $[C] =$ 

⇒ $\boxed{\mathcal{C}_{|U} \text{ has equations } x_i y_i - t_i = 0 \text{ around } q_i}$

Remark: why you should not worry when I say stack

we will never worry about stackyness issues

We will never worry about stackyness issue
and we just take into account automorphisms
to compute dimensions.

This means, that when we talk about
- smoothess at a point
- singularities at a point
- equations at a point

we are talking about properties of the
smooth chart U.

When the automorphisms are finite (DM stacks)
then the chart U will be étale.
(no need to worry
for the dimension either)

↪ deformations of a node C
↪ $M_{g,n}$

§ 2.2 Deformations of (C, L) and deformations of the sections.

Theorem ["Deformation theory" either by Sernesi or by Hartshorne]

- Let C a curve, L a line bundle on it
  the first order deformations of L leaving
  C fixed $\boxed{\mathrm{Def}_L(D_\varepsilon) = \mathrm{Def}_{(C,L)/C} \simeq H^1(C, \mathcal{O}_C)}$
  For C we take the trivial def $C \times D_\varepsilon$
- The obstructions are contained in $H^2(C, \mathcal{O}_C) = 0$
  so there are no obstructions

- The infinitesimal automorphisms are given
  by $H^0(C, \mathcal{O}_C)$ $\mathrm{Aut}(L) = \mathbb{C}^*$

proof 1 idea
$\mathcal{L}$ is flat over D_ε {

$\mathrm{Def}_L(D_\varepsilon) = \{ \mathcal{L} \text{ on } \begin{matrix} C \times D_\varepsilon \\ \downarrow \\ D_\varepsilon \end{matrix} \mid \boxed{\mathcal{L}|_0 = L} \}$

Look at the sequence

$[\, e : 0 \to \mathbb{C} \xrightarrow{\varepsilon} \mathbb{C}[\varepsilon]/_{\varepsilon^2} \to \mathbb{C} \to 0$

$\Rightarrow$ if $\mathcal{L} \in \mathrm{Def}_L(D_\varepsilon)$ (and so is flat over D_ε)

$e \otimes \mathcal{L} : 0 \to L \xrightarrow{\varepsilon} \mathcal{L} \to L \to 0$ $\overset{=}{\text{is exact}}$ $\mathcal{L} \in \mathrm{Def}(D_\varepsilon) \Rightarrow e \otimes \mathcal{L} \in \underset{H^1(C, \mathcal{O}_C)}{\overset{\shortparallel}{\mathrm{Ext}^1_C(\mathcal{L}, \mathcal{L})}}$

$\Leftarrow$ given $e \in \mathrm{Ext}^1(L, L)$ we can define an action of $\underline{\varepsilon}$ on the middle sheaf which makes it into a flat line bundle over $C \times D_\varepsilon$ ▨

Having studing the deformations with C fixed, we now know that the moduli space

$\underline{\mathrm{Pic}^d_{g,n}} = \{ (C, p_1, .. p_n, L) \mid \begin{matrix} - C \text{ prestable} \\ - \deg L = d \end{matrix} \}/\sim$ ← smooth of dim $4g-4+n$

$$\begin{tikzcd} \mathrm{Pic}^d_{g,n} \arrow[d, "p = \text{forget } L"] \\ \mathcal{M}_{g,n} \end{tikzcd}$$

← $\underline{\text{smooth}}$ of $\underline{\text{dim}}$ $\underline{\underset{\dim H^1(C, \mathcal{O}_C) - \dim \mathrm{Aut}}{\overset{\shortparallel}{g-1}}}$

$\mathcal{M}_{g,n}$ ← smooth of dim $3g-3+n$

- $\underline{p \text{ is smooth}}$ (no obst. for def of L with fixed C)
  $\Rightarrow$ $\boxed{\underline{\mathrm{Pic}}^d_{g,n} \text{ is } \underline{\text{smooth}}}$
- $\underline{p \text{ has}}$ dim = $\underbrace{\text{dimension of the relative tg}}_{H^1(C, \mathcal{O}_C)}$ $-$ $\underbrace{\text{dim of the relative automorphism}}_{H^0(C, \mathcal{O}_C)}$
  $= \underline{g \qquad - 1}$

Finally, to $\underline{\text{understand dimension and regularity}}$ of

$\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) = \{ (C, \underline{p}, L, s_0 .. s_r) \mid \begin{matrix} \to s_i \text{ define a map} \\ \to \text{the map is stable} \end{matrix} \}$, we have to understand the map

$$\boxed{\overline{M}_{g,n}(\mathbb{P}^2, d) \xrightarrow[\varsigma]{} \mathrm{Pic}^d_{g,n}}$$ ⇝ we will see tomorrow

One we have (C, L), in order to give a map to $\mathbb{P}^2$ we have to choose $\underline{r+1 \text{ sections of}}$ $\underline{H^0(C, L)}$

$\uparrow$ when C is smooth and $d > 2g-2$ then $H^1(C, L) = 0$ and $\dim H^0(C, L) = d - g + 1$

$\Rightarrow$ we would expect ς to have dimension $(r+1)(d-g+1)$

But $\boxed{\text{is } \varsigma \text{ smooth?}}$ i.e given a deformation of (C, L), do the sections $s_0, \dots s_r$ deform?

$\boxed{\text{NO}}$: the $\boxed{\text{obstruction lie in } H^1(C, L)}$

Recall that for example, without moving C, we had

$$0 \to L \xrightarrow{\varepsilon} \mathcal{L} \to L \to 0$$

$$\Rightarrow \quad \underbrace{H^0(\mathcal{L}) \to H^0(L) \xrightarrow{S} H^1(L)}$$

if $S(s) = 0 \Rightarrow$ the section extends
<u>otherwise NOT</u>!! [see [Wang, "Def. of pairs (X, L)"] for formal treatment

§ <u>Recap</u>

We have

NOT smooth
obs in $H^1(C, L)$

smooth of dim $g-1$

$$\begin{tikzcd} \overline{M}_{g,n}(\mathbb{P}^2, d) \arrow[r, "\varsigma"] & \mathrm{Pic}^d_{g,n} \arrow[r, "p"] & \mathcal{M}_{g,n} \end{tikzcd}$$

smooth of

smooth of dim $3g-3+n$