↳ The blow up produce $\widetilde{M}^{main}_{g,n}(\mathbb{P}^2, d)$ $g = 1, 2$ smooth and birational to $\overline{M}^{main}_{g,n}(\mathbb{P}^2, d)$ ∎

The construction is extraordinary! But

- The procedure is very involved and keeping track of the local structure around any possible type of point requires quite some effort!

- The blow up has (apparently) destroyed the modular meaning.
  For example to show that $\widetilde{M}^{main}_{g,0}(\mathbb{P}^2, d)$ is smooth we really have to look at equations; we do not have a natural def-obs.

Talk 3

§ 6. Good factorization through a Gorenstein curve

The obstruction come from $H^1(L) \neq 0$ when
→ f contract a component of genus 1 and 2
(→ f has degree 2 and so the image is a $\mathbb{P}^1$ in genus 2)

Can't we get rid of these contracted component immediately?
Can't we replace every $f: C \to \mathbb{P}^2$ where a component $Z \subset C$, with $g(Z) = 1, 2$ with an $\bar{f}: \bar{C} \to \mathbb{P}^2$

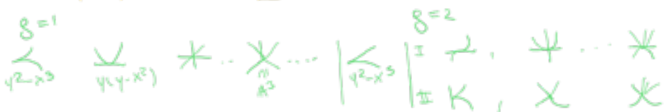
- $\bar{C}$ is obtained contracting the higher genus component
- $\deg \bar{f}$ is high enough on each component, i.e
  $H^1(\bar{C}, \bar{L}) = 0$ $\bar{L} = \bar{f}^* \mathcal{O}(1)$

In other words, can't we consider a moduli space of diagrams

$$(*) \quad \begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\tau"] & \mathbb{P}^2 \\ \bar{C} \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

with
- $Exc(\tau) \subseteq Exc(f)$
- $\bar{f}$ unobstructed
- $\bar{C}$ of genus $= g(C)$ (with singularities we can control!)

Equations and Picture of $\bar{C}$ for low # of branches [Smyth, Battistella]



branches depends on the # of rational tails

1) Given $[f] \in \overline{M}_{g,n}(\mathbb{P}^2, d)$ we CAN'T always find a factorization (*)

   (*) exists <=> $[f]$ is in the closure of $\overline{M}^{main}_{g,0}(\mathbb{P}^2, d)$

   g=1 Vakil-Zinger Ranganathan-Santos-Parker-Wise RSPW BC
   g=2 3

   ↳ go back to comp. picture and draw int with main

2) For $[f] \in \overline{M}^{main}_{g,0}(\mathbb{P}^2, d)$ there does not exist a well defined factorization: (does not work in families)
   there are different $\bar{C}$ one can factor through

→ In genus 1, to give a $\bar{C}$ we need to choose a linear relation $\alpha_1 v_1 + \dots + \alpha_k v_k = 0$

where  $v_1, \dots v_k$ are tangent at $R_1, \dots R_k$ in q_i

→ I genus 2, to give a $\bar{C}$ we need to choose

- special branch(es) R_i (R_j)
- A linear relation among the v_j of the non special branches
- we need to know if the special branch R_i is attached to a Weierstrass point

THE ISSUES SUGGEST THE SOLUTION:

1. Modify the moduli functor in such a way that at the boundary (i.e when there are nodes) there is some extra structure which allow us to construct a universal $\mathcal{C} \to \bar{C}$

2. Takes only the maps admitting a factorization

⇝ This would produce a birational model of $\overline{M}^{main}_{g,0}(\mathbb{P}^r, d)$, which is not obstructed and has a modular interpretation

ADVERTSIMENT:
whether you want to try find a modular interpretation of the blow up or you want the extra structure to construct $\mathcal{C} \to \bar{C}$, Log geometry is going to help!

§7. Log schemes: definition and Examples

Def: Let X be a scheme. A pre-log structure on X is a sheaf of monoids M_X together with $M_X \xrightarrow{\alpha} \mathcal{O}_X$.

α is called "exponentiation" map.

A log structure is a pre-log structure s.t $\alpha^{-1}(\mathcal{O}_X^*) \xrightarrow{\simeq} \mathcal{O}_X^*$

give a log structure we call characteristic sheaf

$$\overline{M}_X = M_X / \mathcal{O}_X^*$$ (include the combinatorial part of the info)

given a pre log structure there is an associated log structure M_X^a defined by

$$\begin{array}{ccc} \alpha^{-1}(\mathcal{O}_X^*) & \longrightarrow & \mathcal{O}_X^* \\ \downarrow & \ulcorner & \downarrow \\ M_X & \longrightarrow & M_X^a \end{array}$$

Example

Let $X = \operatorname{Spec} A$ and P a monoid e.g. $P = \mathbb{N}^k$. Then P determine the log structure

$$M_X =: \mathbb{C}^* \oplus P \longrightarrow \mathbb{C}$$
$$(\lambda, p) \longrightarrow \begin{cases} \lambda & p=0 \\ 0 & p\neq 0 \end{cases}$$

Example 1 : Divisorial log structure

X smooth variety and $D \subseteq X$ a divisor

Define M_X as

$$M_X(U) = \{ f \in \mathcal{O}_X(U) \mid f|_{U\setminus D} \in \Gamma(U\setminus D, \mathcal{O}_X^*) \}$$

↳ sheaf of functions which are invertible outside D

$\Rightarrow$ $\alpha : M_X \longrightarrow \mathcal{O}_X$ is the inclusion

and clearly we have

$$\alpha^{-1}(\mathcal{O}_X^*) = \mathcal{O}_X^* \subseteq M_X$$

Example 2 : standard log structure on affine toric varieties

Let X_P an affine toric variety

we have see that

$X_P \longleftrightarrow \sigma$ a convex cone $\subseteq \mathbb{R}^n$ of max dim $n = \dim X_P$

$X_P = \operatorname{Spec}(\mathbb{C}[P])$ where

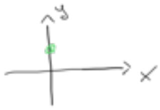
$\boxed{P \text{ is the monoid } \sigma^\vee \cap \underset{\text{character lattice}}{M}}$

$\Rightarrow$ the standard log structure on X_P is the one associated to

$$P \longrightarrow \mathbb{C}[P]$$

Sub-Example :

$X = \mathbb{A}^2 = \operatorname{Spec} \mathbb{C}[\mathbb{N}^2]$



Then

$$\mathbb{N}^2 \xrightarrow{\alpha} \operatorname{Spec} \mathbb{C}[\mathbb{N}^2]$$
$$(a,b) \longrightarrow x^a y^b$$

Let us try to understand $\overline{M}_X$ by looking at its stalks

The definition of M_X as pushout tells us that

$$M_X / \mathcal{O}_X^* = \mathbb{N}^2 / \alpha^{-1}(\mathcal{O}_X^*)$$

- First take $p \in \mathbb{A}^2 \setminus \{x=0\} \cup \{y=0\} := U$

$\Rightarrow$ $x^a y^b \in \mathcal{O}_X^*|_U$ $\forall\, a, b$ $\Rightarrow$ $\alpha^{-1}(\mathcal{O}_X^*|_U) = \mathbb{N}^2|_U$

$\Rightarrow \boxed{\overline{M}_{X,p} = 0}$

- if $p \in \{x=0\} \setminus \{y=0\}$ $\Rightarrow$ $x^a y^b$ is invertible

$\Leftrightarrow$ $a=0$ $\Rightarrow$ $\alpha^{-1}(\mathcal{O}_X^*) = \mathbb{N}_{\langle 0,1\rangle}$

$\Rightarrow$ $\overline{M}_{X,p} \simeq \mathbb{N}$

- Finally if $p = (0,0)$

$\Rightarrow$ $\overline{M}_{X,0} = \mathbb{N}^2$

Rmk. Notice that this is the same as the [illegible] log structure given by functions which are invertible outside $D = \{xy = 0\}$]

<u>Def</u>: A log morphism $(X, M_X) \to (Y, M_Y)$ is a morphism of schemes $X \xrightarrow{f} Y$ togeter with

$$\begin{tikzcd} f^{-1}M_Y \arrow[r] \arrow[d] & M_X \arrow[d] \\ f^{-1}\mathcal{O}_Y \arrow[r] & \mathcal{O}_X \end{tikzcd}$$

we call $f^* M_Y$ the log structure associated to $f^{-1} M_Y \to \mathcal{O}_X$

A log morphism is called strict when $\boxed{M_X = f^* M_Y}$

The example $P \to \operatorname{Spec} \mathbb{C}[P]$ is the most important one for the following reason:

given a log scheme (X, M_X), for each $x \in X$, $\exists$ a (étale) neighborhood U_x and a monoid P_x and a map

$U_x \xrightarrow{\varphi} \operatorname{Spec} \mathbb{C}[P_x]$ s.t.
$M_X|_{U_x}$ is $\varphi^* (P_x \to \mathbb{C}[P_x])^a$] chart for M_X on U_x

We are only interested in log schemes where the local P_x:

1. Are finitely generated
2. $P_x \hookrightarrow P_x^{gp}$ is injective
3. if $n \cdot e \in P_x$ $\exists\, n \in \mathbb{N} \Rightarrow e \in P_x$

] this are satisfied in the case of toric variety

you should think that $P_x = \overline{M}_{X,x}$

§ 7.5 Tropicalization

Something nice about toric variety is that lot of computation / geometric operations can be done / understood combinatorially, i.e at the level of the Fan.

This principle admit a generalization to log schemes:

given $(X, M_X) \underset{\text{tropicalization}}{\rightsquigarrow} \Sigma(X)$

$\Sigma(X)$ is a cone complex, i.e. is a union of cones glued along faces
$(\sigma_{\mathbb{R}}^{\vee} \subset \Lambda \otimes \mathbb{R}, \underset{\text{lattice}}{\Lambda})$

key difference:

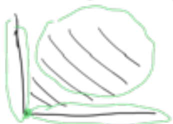
while a Fan admits a global embedding
$\Sigma \subseteq \Lambda \otimes \mathbb{R}$ in a lattice of dim = dim X

you can think that $\Sigma(X)$ is obtained by (X, M_X) taking as cones

$$\left((\overline{M}_{X,\eta})^{\vee}_{\mathbb{R}}, (\overline{M}_{X,\eta})^{*} \right)$$ where η is the generic point of a strata where the characteristic sheaf is constant and gluing them accordingly to the generization maps

Example 0

If we look at the tropicalization for the example $\mathbb{N}^2 \to \mathrm{Spec}\,\mathbb{C}[\mathbb{N}^2]$ before we find the Fan of $\mathbb{A}^2$



In general for toric variety we find their Fan

Example 1

X smooth $D \subseteq X$ smooth divisor and M_X the divisorial log structure

$\Rightarrow \overline{M}_X$ has stalk 0 outside D
and stalk $\mathbb{N}$ on D (generated by f^n)
f local eq for D

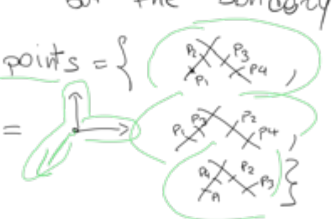
$\Rightarrow Z(X) =$

$\mathbb{R}$

Example 2 $M_{0,4}$ =

Consider $\overline{M}_{0,4} = \mathbb{P}^1$ but the boundary D is naturally 3 points = {

$\Rightarrow \mathrm{Trop}(\overline{M}_{0,4}, M_D) =$



yesterday:

- we need to construct s.t wt of the higher genus subcurves of $\overline{C}$ is high enough

$C \to \overline{C}$
$\downarrow$
$\widetilde{m} \to m^{wt}_{1,n}$
$3,n$

Warning!

- We saw def of log schemes and their tropicalizations

$U_X \xrightarrow{f} \mathrm{Spec}(\mathbb{C}[P_X])$ φ strict but not étale, or smooth

Talk 4