- we need to construct $\mathcal{M} \to \mathcal{M}_{g,n}$ s.t wt of the higher genus subcurves of $\overline{C}$ is high enough

Warning!

- We saw def of log schemes and their tropicalizations

$$U_x \xrightarrow{\varphi} \operatorname{Spec}(\mathbb{C}[P_x])$$

φ strict but not étale, or smooth

Talk 4

§7. Log-Blow-ups and their modular meaning

Given a log scheme (X, M_X) and a log ideal $\langle x_1,\dots,x_m\rangle = K \subseteq M_X \xrightarrow{\alpha} \mathcal{O}_X$ we have:

- $X_K = Bl_{\mathcal{I}(K)} X \xrightarrow{p} X$

- Locally $\overline{M}_{X_K}$ is given by $\langle p^*\overline{M}_X, \kappa - \kappa_i, \forall \kappa \in K\rangle$

Blow-up as ordering.

Recall that given a monoid $\overline{M}$ there is a partial order on its elements:

$a > b$ if $a - b \in \overline{M} \subset \overline{M}^{gp}$

Say we consider a log ideal with 2 generators $K = \langle a, b\rangle \subset M$

$\Rightarrow$ In the log blow-up $Bl_K U$ we have two charts

U_1, U_2 where $\overline{M}_K|_{U_1} = \overline{M}[a-b]$, $\overline{M}_K|_{U_2} = \overline{M}[b-a]$

And these are glue along the open locus $a - b = 0 \in \overline{M}$.

$\boxed{\text{Hence in the log blow-up } a \text{ and } b \text{ are ordered!}}$

Example

$X = \mathbb{A}^2$, $M_X = (\mathbb{N}^2 \to \mathbb{C}[\mathbb{N}^2])^a$

Let us take the log ideal generated by $\mathbb{N}^2 = \langle (1,0), (0,1)\rangle \subseteq M_X$

$\mathbb{P}^1 \times \mathbb{A}^2$

$\cup | \, x t_1 - y t_0 = 0$

$\Rightarrow X_K = Bl_0 X$

$\overline{M}_{X_K}|_{U_{\langle(1,0)\rangle}} = \langle \overset{a}{(1,0)}, \cancel{(0,1)}, \overset{b-a}{(-1,1)}\rangle$

$|_{U_{\langle 0,1\rangle}} = \langle b, a-b\rangle$

And $\operatorname{Trop}(X_K, M_{X_K})$

In general if $K = \langle \alpha_1,\dots,\alpha_m\rangle \subset M$, then we take the log blow-up (X_K, M_{X_K}), this will have the property that locally there is a minimal element of the monoid.

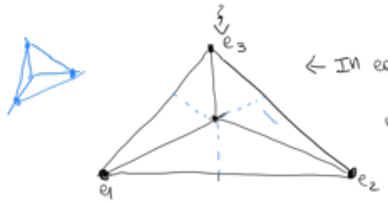
This correspond to the local principal generator of $p^{-1}(\alpha(K))$, which is the ideal sheaf of the exceptional divisor.

Example

Take $\mathbb{A}^3$ with its toric log structure

Let consider the $K \subseteq \overline{M}$
$\langle e_1, e_2, e_3 \rangle$

$\Rightarrow Bl_K \mathbb{A}^3 = (Bl_0 \mathbb{A}^3, A_x + A_y + A_z + E)$



← In each cone we have a minimal element

you can further blow up e_1+e_3, e_2+e_3, e_1+e_2 to obtain the total ordering.

Lemma (Santos-Parker)

Let (S, M_S) a log scheme,

$K \subseteq M_S$ a finite collection of sections

$\Rightarrow$ there exist a log scheme $(Ord_K(S), M_{S_K})$ and a log morphism

$(Ord_K(S), M_{S_K}) \longrightarrow (S, M_S)$ which is a sequence of log blow-ups

such that the map

$\overline{M}_S \longrightarrow \overline{M}_{S_K}$

sends elements of $\overline{K}$ to ordered elements in $\overline{M}_{S_K}$

This simple observation is what makes log geometry and log blow-ups so well suited for modular desingularization!

§8. Log & tropical curves

Given a log scheme (S, M_S)

Def: a family of log-smooth curves over S is a log-smooth, proper map

$(C, M_C) \xrightarrow{\pi} (S, M_S)$

$$\begin{array}{ccc} (T', M_{T'}) & \to & (X, M_X) \\ \downarrow \varphi & & \downarrow \\ (T, M_T) & \to & (Y, M_Y) \end{array}$$

with 1-dim'l connected Fibers, s.t · φ strict φ exact.
$\underbrace{\pi \text{ is integral \& saturated}}$

↳ technical conditions on the morphism at the level of monoids ensuring that

π is flat with reduced Fibers

Theorem [Kato]

Log smooth curves have at worst nodal singularities.

Theorem [Kato]: characterization

Let $(C, M_C) \to (S, M_S)$ a family of log-smooth curves.

let $x \in C$ a point; there exist

étale neighborhood

$$\begin{tikzcd} x \in U_V \arrow[r, hook] \arrow[d, "\pi"] \arrow[dr, phantom, "\square"] & C \arrow[d] \\ \pi(x) \in V \arrow[r] & S \end{tikzcd}$$

s.t:

1) If x is a smooth point $\Rightarrow U_V = \mathcal{O}_V[s] = \mathbb{A}^1_V \to V$ and the log structure is pulled back from the base

$$\text{so } \overline{M}_{C,x} = \overline{M}_{S,\pi(x)}$$

2) If x is a marked point $\Rightarrow U_V = \mathbb{A}^1_V \to V$ and the log structure is generated by $\pi^* M_S$ and the 0 section

$$\text{so } \overline{M}_{C,x} = \overline{M}_{S,\pi(x)} \oplus \mathbb{N}$$

3) If x is a node the $U_V = \mathcal{O}_V[x,y]/xy-f$ for some $f \in \mathcal{O}_V$

$$\overline{M}_{C,x} = \{(p,p') \in \overline{M}^{\oplus 2}_{S,\pi(x)} \mid p-p' \in \mathbb{Z}S_x\}$$

where $S_x \in \overline{M}_{S,\pi(x)}$ is the smoothing parameter

[there is a lift $\tilde{S}_x \in M_{S,x}$ s.t $\alpha(\tilde{S}_x) = f$]

Important Remark

For any log smooth curve $\underline{C} \to \underline{S}$ there exist a minimal / canonical log structure M_S^{min} such that any $(C, M_C) \to (S, M_S)$ is pulled back from $(C, M_C^{min}) \to (S, M_S^{min})$

In the case $S = pt$, then

$$\overline{M}_S^{min} = \mathbb{N}^{\# \text{nodes of } C} = |E(\Gamma(C))|$$

The canonical log structure above is the one is the one coming from the divisorial log structure on $\overline{\mathcal{M}}_{g,n}$: $(\overline{\mathcal{M}}_{g,n}, M_{\partial_1})$

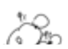
where ∂_1 is the boundary divisor

∂_1 = locus of curves with at least one node

$\{t_1 \cdot t_2 \cdots t_k = 0\}$

$$\partial_1 \cap U[C] \text{ is } \bigcup_{i=0}^{k} \{ t_i = 0 \}$$

around a point

$[C] =$ 

where we recall

$U_{[C]}$ looks like $\mathbb{A}^{3g-3+n} \times \mathbb{A}_{t_1,\dots,t_k}$ smoothing parameters of the nodes

depends from the cpx moduli, i.e. from (g,n)

$\rightsquigarrow$ $\mathrm{Trop}(\mathcal{M}_{g,n}, \mathcal{M}_{\mathcal{O}_\cdot})$ locally around $[C]$ is given by $\mathbb{N}^{\oplus k} \otimes \mathbb{R}_{\geq 0}$ where

k = # nodes

<u>Def</u> A pre-stable, n-marked <u>tropical curve</u> Γ is a finite graph $\Gamma = (V, E)$ together with

(1) a marking funct $m: \{1, \dots, n\} \to V$

(2) a genus funct. $g: V \to \mathbb{N}$

(3) A lenght function $\ell: E \to \mathbb{R}_+$ (E = set of bounded edges)

For σ a cone, a family of tropical curve over σ is a tropical curve $\Gamma = (V, E, m, g, \ell)$ as before with the difference that

$$\boxed{\ell: E \longrightarrow S_\sigma \setminus \{0\} \quad \leftarrow \text{monoid dual to } \sigma}$$

$S_\sigma = \sigma^\vee \cap M$

Given a log smooth curve $(C, M_C) \to (S, M_S)$ as before

there is an associated family of tropical curve.

If <u>$S = pt$</u> $\Rightarrow$ <u>Γ is the dual graph of C</u> ($\Gamma = (V, E, m, g)$)

and $\ell: E \longrightarrow \overline{M}_S$

$e \longrightarrow s_e = \log(t) =$ "smoothing parameter"

§ 8.5 Line bundles on log curves

We have seen that given a log scheme (X, M_X) we have a s.e.s

$$0 \to \mathcal{O}_X^* \to M_X^{gp} \to \overline{M}_X^{gp} \to 0$$

$\Rightarrow$ we have a map

$$H^0(X, \overline{M}_X^{gp}) \longrightarrow H^1(X, \mathcal{O}_X^*) \cong \mathrm{Pic}(X)$$

<u>Lemma</u>

For $C \to S$ a log smooth curve

$H^0(C, \overline{M}_C^{gp}) =$ Piecewise linear Functions on C, i.e

- For each $v \in \Gamma$, $\lambda(v) \in \overline{M}_S$
- For each leg (or infinite edge) h, an $n_h \in \mathbb{N}$
- λ has integer slopes along bounded edges

$\downarrow$

i.e $v_1 \xrightarrow{e} v_2$

$$\boxed{\mu_e(\lambda) = : \frac{\lambda(v_2) - \lambda(v_1)}{s_e} \in \mathbb{Z}}$$

slope outgoing from v_1

Proposition [RSPW]

$C \to S$ log smooth curve; over a geometric point $s \in S$, every vertex $v \in \Gamma$ correspond to a component C_v of C_s and we have:

$$\boxed{\mathcal{O}_C(\lambda)\big|_{C_v} = \mathcal{O}_{C_v}\Big(\sum_{e \to v} \mu_\lambda(e)\, p_e\Big)}$$

We are going to use these piecewise linear function to construct the line bundle $\mathcal{M}$ on $\mathcal{C}$ giving the contraction.

§3: Facts from birational geometry

- Given a $\mathcal{C} \xrightarrow{\pi} S$ with total space of $\mathcal{C}$ smooth and a line bundle $\mathcal{M}$ which is π-semiample and $\pi_* \mathcal{M}^{\otimes n}$ is a vector bundle for $n \gg 0$

$\Downarrow$

There exist a contraction

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\varphi"] \arrow[dr, "\pi"'] & \overline{\mathcal{C}} = \mathrm{Proj}_S\big(\bigoplus_n \pi_* \mathcal{M}^{\otimes n}\big) \arrow[d, "\overline{\pi}"] \\ & S \end{tikzcd}$$

s.t:

- $\overline{\mathcal{C}}$ is normal with reduced fibers
- $\overline{\pi}$ flat
- φ is birational

$\mathcal{C}$
$\downarrow$
$\overline{\mathcal{C}}$

Moreover, $\overline{\mathcal{C}}$ is Gorenstein

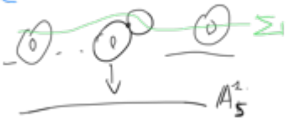
$\Updownarrow$

there exist a π-semiample $\mathcal{M}$ s.t

$$\mathcal{M} = \omega_{\mathcal{C}/S}(D)(\Sigma)$$

where
- $D = \sum_{F \in \mathrm{Ex}(\varphi)} n(F) \cdot F$, $n(F) > 0$
- Σ is a divisor not intersecting the contracted locus

Example / Exercise

Let $\mathcal{C} \to S$ = 

Can we find M like we asked before?

Recall:

- ADJUNCTION FORMULA ON SURFACES: $Z \overset{\text{comp.}}{\subseteq} \overset{\text{fiber}}{C_s}$

$$\omega_{\mathcal{C}/S}|_Z = \omega_Z(-Z^2)$$

- To compute Z^2 recall that for a fiber C_s

$$(C_s)^2 = 0$$

- For a genus 1 nodal curve Z with no rational tails attached $\omega_Z = \mathcal{O}_Z$.

and $\omega_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$

- how do the fibers of the contraction look like

$$\begin{tikzcd} & \mathcal{C} \arrow[dl] \arrow[dr] & \\ \overline{\mathcal{C}}_1 \arrow[rr] & & \overline{\mathcal{C}}_2 \end{tikzcd}$$

$\overline{\mathcal{C}}_1 = \mathrm{Proj}(Z_1)$ $\quad$ $\mathrm{Proj}(Z_2)$

Bir. Rigity Lemma Debarre

Talk 5

§ 10. One parameter smoothings

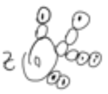
If $\mathcal{C} \to \Delta_S$ (with sections $\sigma_1, \dots, \sigma_n$) is a one parameter smoothing (with smooth total space)

↳ around each node eq is $xy - s$

of a genus 1, genus 2 curve. Suppose we want to contract the components which are no marked.

Q: $M \simeq \omega_{\mathcal{C}/\Delta_S}(D + Z)$ to perform the contraction?

→ in $g = 1$ the answer is yes when looking at the



$\boxed{\text{closest marked component to } Z \text{ are all at the same distance from } Z}$