$[\,\textcircled{0},\ \alpha,\ X_{\cdots}\,]$ and $\omega_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$

- how do the fibers of the contraction look like

$$\begin{array}{ccc} & e & \\ \swarrow & & \searrow \\ \bar{e}_1 & \longrightarrow & \bar{e}_2 \\ \text{Proj}(Z_1) & & \text{Proj}(Z_2) \end{array}$$

Bir. Rigity Lemme Debarre

Yesterday:

- we saw that log blowups $\longleftrightarrow$ introduce ordering among elements in the monoid $\updownarrow$ subdivision of the tropicalization
- Log smooth curves and $(\mathcal{M}_{g,n}, M_{\mathcal{O}_1})$
- Tropical curves and e.b. from $\lambda \in PL(\mathcal{C} \to \overline{M}_S)$
- want $\boxed{\mathcal{M} = \omega_{e/S}(D + Z)}$ + globally generated $/S$

supported on the contracted locus

ample outside the contract. locus

Talk 5

§10. One parameter smoothings

if we drop smoothness we could choose $xy - s^3$

If $\mathcal{C}$ is a one parameter smoothing (with smooth total space) $\hookrightarrow$ around each node eq is $\boxed{xy - s}$

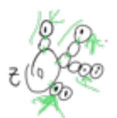
$\sigma_i, \circlearrowleft$ $\downarrow$

Δ_S σ_i's

of a genus 1, genus 2 curve. Suppose we want to contract the components which are no marked in the central fiber

Q: When can we find $M \simeq \omega_{e/\Delta_S}(D + Z)$ to perform the contraction?

→ in $g = 1$ the answer is yes when looking at co the



closest marked component to Z along the various branches are at same distance from Z

Z = minimal $g = 1$ subcurve

Fix: blow up the marked pts which are to close till ☐ is satisfied and then contract

$$\omega_{e/S}(D + \Sigma)$$

→ In $g = 2$, the answer is yes when

or

- the special branch is the closest to Z'
- Along all the other branches the first marked epts occur at the same distance

$$\left[\begin{array}{lll} d\,\text{tail} = 2\,d_{\text{special}} & & NW \\ \phantom{d\,\text{tail}} = 3\,d_{\text{special}} & & W \end{array}\right.$$

Draw these condition tropically

$g=1$

$g=2$

has radius
- d along the special branch
- 2d
3d on the other

$\boxed{\text{we need to be able to talk about distances and compare them!}}$

tropical curves — log blow-ups, ordering

minime = no rational tail attached C_0 = $g=1$ 0, $g=0$

§ 9. Genus 1 resolution [RSPW]

Def: given $C \to S$ a log smooth curve of genus 1, with $S = \operatorname{Spec} \sigma$. Let Γ its tropicalization and $S_e \in \overline{M}_S$ the "lenght" of the edge $e \in \Gamma$.

Let $\Gamma_0 \subset \Gamma$ the $\underbrace{\text{minimal } g=1 \text{ subcurve}}_{\text{we call it core}}$ $\Rightarrow \exists\, v \in \Gamma \setminus \Gamma_0$

$\exists!$ path $e_1 \dots e_k$ from v to Γ_0

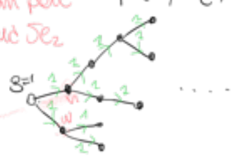
define $\boxed{\lambda(v) = \sum_{i=1}^{k} S_{e_i}}$ = distance from v to the core

Notice that $\lambda \in PL(\Gamma; \overline{M}_S) \rightsquigarrow$ gives a line bundle! on C

$$\overset{\shortparallel}{\Gamma(C, \overline{M}_C)}$$

we are asking to compare S_{e_1} and S_{e_2}

$v_1 \xrightarrow{e} v_2$



I will just draw

$\psi_\lambda(e)$ = slope of λ $= \dfrac{\lambda(v_2) - \lambda(v_1)}{S_e}$

Lemma: λ is compatible with edge contractions (smoothings). For any base scheme S

$\Rightarrow$ given $\boxed{C \xrightarrow{\pi} S}$ log smooth $\exists$ $\boxed{\lambda \in \Gamma(C, \overline{M}_C) = \Gamma(S, \pi_* \overline{M}_C)}$ extending the function λ of the previus definition to a global section. $\forall s \in S$ λ_s is the one of the definition above

Def we say that a log smooth genus one curve $C \to S$ is radially aligned if for all $s \in S$, $\forall\, v, w \in \Gamma_s \setminus \Gamma_{0,s}$ $\boxed{\lambda_s(v) \text{ and } \lambda_s(w) \text{ are comparable}}$

Proposition

- There exist an algebraic log stack $\left(\mathcal{M}^{rad}_{1,n}, M_{\mathcal{M}^{rad}_{1,n}}\right)$ parametrizing radially aligned curves
- $\left(\mathcal{M}^{rad}_{1,n}, M_{\mathcal{M}^{rad}_{1,n}}\right) \to \left(\mathcal{M}_{1,n}, M_{\mathcal{M}_{1,n}}\right)$ is a log modification

  a sequence of log blowups
- $M_{\mathcal{M}^{rad}_{1,n}}$ is locally generated by $M_{\mathcal{M}_{1,n}}$ and by the differences $\lambda_s(v) - \lambda_s(w)$

  $\overset{\shortparallel}{M_{\Theta_1}}$

Claim! $\left(\mathcal{M}^{rad}_{1,n}, M^{rad}_{\mathcal{M}_{1,n}}\right)$ is the $\widetilde{\mathcal{M}}$ on which we will construct the universal contraction!

C can be 

Fact: Let C a genus 1 Gorenstein curve with no tail attached. A line ble L of $\deg > 0$ has $\boxed{H^1(C, L) = 0}$

so if on $\overset{0}{\underset{0}{\ast}}$ d with $d>0$ ☺

λ_s = distance to the core of the closest vertex of positive weight

Def: let $C \to S$ a family of radially aligned weighted curve. For $C_s \to s$ define

$$\boxed{r_s = \min_{v \mid wt(v) > 0} \lambda(v)}$$

radius of the circle!

The r_s glue to a global section $\boxed{r \in \Gamma(S, \overline{M}_S)}$

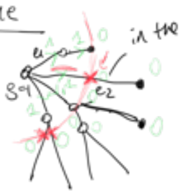
Proposition: given a radially aligned $\mathcal{C} \to S$ there exist a birational modification (a blow up)

$$\begin{array}{c} \widetilde{\mathcal{C}} \\ \downarrow \scriptstyle{\rho} \\ \mathcal{C} \\ \downarrow \\ S \end{array}$$

such that on $\widetilde{\mathcal{C}}$ we can define a piece-wise linear function

$$\boxed{y = \max\{r - \lambda, 0\}}$$

Picture



in the monoid we have $s_{e_2} - s_{e_1}$

$\circ$ = wt 0

$\bullet$ = wt >0

$\times$ = a new vertex of genus 0

consider the line bundle on $\widetilde{\mathcal{C}}$ defined by

$$\boxed{\omega_\pi(y)(2\Sigma_1) := M}$$

where Σ_1 is supported on the weight > 0 epts

To be completely precise:

$\leadsto \Sigma$ on $\tilde{C}$ only exist locally

Notice that

i.e. interior of circle

$$0 \to \mathcal{O}_C(-\psi) \to \mathcal{O}_C \to \mathcal{O}_{\mathrm{supp}(\psi)} \to 0$$

closed locus where $\psi \neq 0$

So twisting by ψ means twisting by an effective divisor supported on the contracted locus!

Exercise

- Using the l.b. description of Lecture 4, determine the value of ψ on each vertex of the curve above
- Prove that $M|_{C_v}=0$ for each v in the strict interior of the circle, and has $\deg > 0$ otherwise

Theorem On $\mathcal{M}^{red-wt}_{1,n}$ we have

$$\begin{tikzcd} & \tilde{C} \arrow[dl, "\rho"'] \arrow[dr] & \\ C \arrow[rr, dashed] \arrow[d] & & \overline{C} = \mathrm{Proj}(\pi_* M) \arrow[dll] \\ \mathcal{M}^{red-wt}_{1,n} & & \end{tikzcd}$$

s.t. $\overline{C}$ is Gorenstein and $\boxed{wt(\overline{C}_0) > 0}$

proof:

① prove the statement on the fibres
- i.e. prove that M is semiample ‖ Trick
- $\pi_* M^{\otimes n}$ is e.f. $n \gg 0$

② argue that the local construction glue [Birational rigidity]

Theorem: Let $VZ_{1,n}(X;\beta)$ be the space parametrizing:
- A family of radially aligned $(C, M_C) \to (S, M_S)$
- A stable map $f: C \to X$ satisfying factorization:

$$\begin{tikzcd} & \tilde{C} \arrow[dl] \arrow[dr] & \\ C \arrow[dr] & & \overline{C} \arrow[dl, "\bar f"] \\ & X & \end{tikzcd}$$

where

$\Rightarrow VZ_{1,n}(X;\beta)$ is proper $\forall X$ proper

and for $X = \mathbb{P}^n$ is unobstructed ■

obstructions are in $\boxed{H^1(\overline{C}, \overline{L})}$ $\bar f^* \mathcal{O}(1)$

§ 11. Genus 2 resolution [Battistella, -]

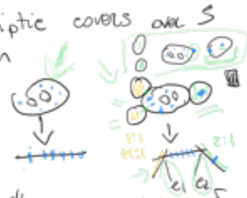
In genus 2, whether one is thinking to interpret the blow-up in a modular way or one has in mind the genus 2 singularities, we see immediately a problem:

In $(\mathcal{M}^{wt}_{2,n}, M_{\Theta_1})$ there is nothing which has to do with Weierstrass or conjugate points !

Step 0: Enrich the log structure

Def: A family of hyperelliptic covers over S is a finite morphism

$$\boxed{\begin{tikzcd} \psi : \mathcal{C} \arrow[r, "2:1"] \arrow[dr] & T \arrow[d] \\ & S \end{tikzcd}}$$



with

- $\mathcal{C}$ prestable of $g=2$, $D^{\psi}_{\mathcal{C}}$ is the divisor of Weierstrass pts (smooth pts where the map branches)
- T is prest $g=0$, D^{ψ}_{T} is the branch div
- locally $\mathcal{O}_S[u,v]/(uv-s) \longrightarrow \mathcal{O}_S[x,y]/(xy-t)$ $\qquad$ $u \to x^i$, $v \to y^i$, $s \to t^i$, $i=1,2$ $(*)$

[Eisenbud-Harris]
Theorem: There exist a smooth stack $\mathcal{A}_2$ par. hyperelliptic 2:1 covers.

There is a minimal log structure defined by

$$\boxed{M^{\psi\text{-min}}_S = M^{\mathcal{C}\text{-min}}_S \oplus_{\mathcal{O}^*_S} M^{T\text{-min}}_S \Big/ \sim \; \frac{\boxed{(0, S_q) \sim (i S_p, 0)}}{i \text{ is the one of}}}$$

Example

$\overline{M}^{\psi\text{-min}}_{S_{pt}} = \langle Se_1, Se_2' \rangle$ where $\boxed{Se_2 = 2Se_2'}$

We can decorate admissible covers with marking, and weights, and impose stabilities

Definition

We consider $\mathcal{A}^{wt\text{-}st}_{2,n}$ where markings and weights are defined on the source and induce markings and weight on T

stability: T must be weighted stable with the weight + markings coming from $\mathcal{C}$.

Fact: We have a birational map

$$\begin{tikzcd} & \boxed{\mathcal{A}^{wt\text{-}st}_{2,n} \arrow[r, "\varphi"]} & \mathcal{M}^{wt\text{-}st}_{2,n} \\ T \arrow[ur] & \mathcal{C} \arrow[l] \arrow[u] \arrow[r] & \mathcal{C}' \arrow[u] \end{tikzcd}$$

blow-down some rational components

not a log blow-up but "we got lucky" with the modular interpretation.

Short version of the story

Theorem:

There exist a log-modification

There exist a log ...

$$\mathcal{A}^{\lambda,wt}_{2,n} \to \mathcal{A}^{wt-st}_{2,n}$$

such that on $\mathcal{A}^{\lambda,wt}_{2,n}$ we can construct

$$\boxed{\begin{tikzcd} & \widetilde{C} \arrow[dl] \arrow[dr] & \\ C \arrow[d] & & \overline{C} \arrow[dl, "\overline{\pi}"] \\ \mathcal{A}^{\lambda w}_{2,n} & & \end{tikzcd}}$$

such that $\overline{C}$ is Flat and Gorenstein and $wt(\overline{C}_0)$ (where $\overline{C_0}$ is the minimal $g=2$ subcurve) is high enough!

$A^{\lambda-wt}_{2,n}$ parametrizes log admissible covers whose vertices in the tropicalization are *aligned* w.r.t a certain function

$$\boxed{\lambda: \quad C \to \overline{M}_{A^{\lambda wt}_{2,n}}}$$

Warning!: λ is not like the one of $g=1$ case!

Theorem The moduli space $VZ_{2,n}(X,\beta)$ parametrizing

- λ-aligned admissible covers $\begin{tikzcd} C \arrow[r] \arrow[d] & T \\ S \arrow[ur] & \end{tikzcd}$
- stable maps $C' \to X$ satisfying the Fact. property

$$\begin{tikzcd} & & \widetilde{C} \arrow[dl] \arrow[ddr] & \\ & C \arrow[dl, "="'] & & \\ C' \arrow[dr, "f"'] & & & \overline{C} \arrow[dll, "\overline{f}"] \\ & X & & \end{tikzcd}$$

is a proper DM stack

which is unobstructed when $X = \mathbb{P}^n$;

$$\boxed{VZ_{2,n}(X,\beta) \xrightarrow[\text{birational}]{} \overline{M}^{main}_{2,n}(X,\beta)}$$

Important things I am omitting

(1) I did not say what λ is; it is not the same as in $g=1$; is related to tropical canonical divisors

(2) In $\overline{C}$ there can be non reduced singularities; ribbons

these allow us to take care of the comp where $f^*\mathcal{O}(1)|_Z = \omega_Z$

(3) $\overline{C}$ is not a Proj, there is a two process to construct it