Q. what's the relation between $\underline{D^b(A)}$ and $D^b(A^\#)$?

If $A^\#$ is obtained by tilting at a torsion pair (T, F) in A and (T, F) is cotilting which means for $\forall E \in A$, $\exists F \in F$ and an $F \twoheadrightarrow E$, then $D^b(A) \cong D^b(A^\#)$.

<u>slicing</u> on D : $\{P(\phi)\}_{\phi \in \mathbb{R}}$

(1) $P(\phi+1) = P(\phi)[1]$ (2) $\mathrm{Hom}(P(\phi_1), P(\phi_2)) = 0$ $\phi_1 > \phi_2$.

(3) $E \in D$, then exists a HN filtration.

. we call objects in $P(\phi)$ semistable of phase ϕ.

<u>EX. 6</u> the phases ϕ_i in the HN filtration of any object $E \in D$ is unique.

$\phi_P^+(E) := \phi_1$, $\phi_P^-(E) := \phi_n$

Let $\mathcal{A} = P(0,1]$ = full extension-closed subcategory generated by all $P(\phi)$ for $\phi \in (0,1]$

$$= \{ E \in \mathcal{D} : \phi_P^+(E) \leq 1 \text{ and } \phi_P^-(E) > 0 \}.$$

Thm $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}$.

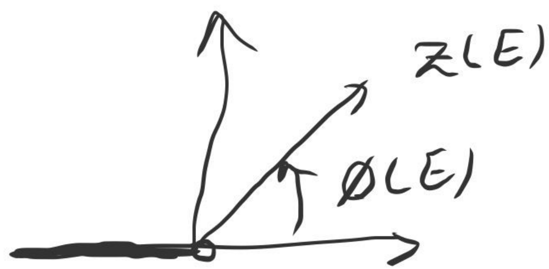
<u>Def</u>. A Bridgeland stability condition on a triangulated category $\mathcal{D}$ is a pair (Z, P) where $Z : K(\mathcal{D}) \longrightarrow \mathbb{C}$ group homomorphism. (called central charge) and P is a slicing.

So that for every $0 \neq E \in P(\phi)$ we have

$$Z(E) = m(E) \exp(i\pi\phi) \quad \text{for some } m(E) \in \mathbb{R}_{>0}.$$

Def. A stability function on an abelian category $\mathcal{A}$ is a group homomorphism $Z_{\mathcal{A}} : K(\mathcal{A}) \longrightarrow \mathbb{C}$ s.t. $0 \neq E \in \mathcal{A}$,

$Z_A(E) \in \mathbb{H} := \{ m \exp(i\pi\phi) : m > 0 , 0 < \phi \leq 1 \} \subseteq \mathbb{C}$



We say $0 \neq E \in A$ is (semi) stable w.r.t Z_A if every $0 \neq F \subset E$ in A, $\phi(F) < (\leq) \phi(E)$

Def. A Harder-Narasimhan filtration of $0 \neq E \in \mathcal{A}$ is a finite filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

whose E_i/E_{i-1} are semistable w.r.t Z_A with

$$\phi(E_1/E_0) > \phi(E_2/E_1) > \cdots$$

Prop. To give a Bridgeland stability condition (Z, P) on D is equivalent to give a pair (A, Z_A) where A is the heart of a bounded t-structure on D and Z_A is a stability function which satisfies HN property.

Pf. $(Z, P) \rightsquigarrow (A, Z_A)$.

$A := P(0,1]$ $\rightsquigarrow$ heart $\Rightarrow K(D) = K(A)$

so the central charge defines $Z_A := Z : K(A) \to \mathbb{C}$.

$0 \neq E \in A$, $0 < \phi(Z(E)) \leq 1$ $\rightsquigarrow$ $Z_A(E) \in \mathbb{H}$

. $(A, Z_A) \rightsquigarrow (Z, P)$

for $\phi \in (0,1]$ $P(\phi) = \{ Z_A\text{-semistable objects in } A \text{ of phase } \phi \}$.

$P(\phi + n) = P(\phi)[n]$ $n \in \mathbb{Z}$.

- Hom vanishing : definition of heart $\mathcal{A}$ + $E, F \in \mathcal{A}$ which are $Z_{\mathcal{A}}$-semistable $\phi(E) > \phi(F)$

  $\mathrm{Hom}(E, F) = 0$

- <u>filtration</u> : $\forall E \in \mathcal{D}$ $\exists$ filtration by cohomology objects $A_i \in \mathcal{A}[k_i]$

$$0 \longrightarrow A_{i,1} \longrightarrow A_{i,2} \hookrightarrow \cdots \hookrightarrow A_{i,m_i} = A_i$$

which is HN filtration of A_i w.r.t $Z_{\mathcal{A}}$.

$$Z := Z_{\mathcal{A}} : \underset{\substack{\| \\ K(\mathcal{A})}}{K(\mathcal{D})} \longrightarrow \mathbb{C}$$

and by our construction Z is compatible with $\mathcal{P}$.

$\square$

$\Rightarrow$ semistable objects are the same w.r.t ~~these~~ two notions.

<u>Example</u>. C: smooth projective curve $/\mathbb{C}$ $\rightsquigarrow$ $\mathcal{A} = \mathrm{Coh}(C)$

$$Z_{\mathcal{A}}(E) = -\deg(E) + i\, rk(E)$$

$\rightsquigarrow (\mathcal{A}, Z_{\mathcal{A}})$ is a stability condition on $D^b(C)$

Semistable objects in $D^b(C)$: shift of μ-semistable vector bundles on C, or zero-dimensional torsion sheaves.

Ex. 7 S: smooth projective surface, then there is no stability function Z which sends objects of $\mathcal{A} = \mathrm{coh}(S)$ to $\mathbb{H}$.

Space of stability conditions.

We need to restrict to $\sigma = (Z, P)$ satisfying two extra assumptions:

(1) Λ: finite dim. lattice with a map $\lambda : K(D) \longrightarrow \Lambda$

s.t.
$$\begin{tikzcd} K(D) \arrow[rr, "Z"] \arrow[dr, "\lambda"'] & & \mathbb{C} \\ & \Lambda \arrow[ur, "Z_\lambda"'] & \end{tikzcd}$$

usually $\Lambda = K_{num}(D)$: quotient of $K(D)$ by the null space of Euler form

$\chi(E, F) = \sum_i \dim \mathrm{Hom}(E, F[i])$

(2) Let $\|\cdot\|$ arbitrary fixed norm on $\Lambda_{\mathbb{R}} = \Lambda \otimes \mathbb{R}$, we assume σ satisfies the "support property",

$$\inf \left\{ \frac{|Z(E)|}{\|[E]\|} \;\middle|\; E \text{ is } \sigma\text{-semistable} \right\} > 0$$

$\rightsquigarrow$ $\mathrm{Stab}_\Lambda(D)$, the set of stability condition on D satisfying the above two condition.

$$\begin{cases} \sigma_1 = (Z_1, P_1) \\ \sigma_2 = (Z_2, P_2) \end{cases}$$

generalised metric on the set of slicing

$$d_{\text{slicing}}(P_1, P_2) = \sup_{0 \neq E \in D} \left\{ |\phi_{P_2}^+(E) - \phi_{P_1}^+(E)|, \; |\phi_{P_2}^-(E) - \phi_{P_1}^-(E)| \right\} \in [0, +\infty].$$

combined this metric with metric on $\Lambda^\vee := \mathrm{Hom}(\Lambda, \mathbb{C})$ induced by $\|\cdot\|$

$\rightsquigarrow$ we get a generalised metric on $\mathrm{Stab}_\Lambda(D)$:

$$d(\sigma_1, \sigma_2) = \sup\{ d_{\text{slicing}}(P_1, P_2), \|Z_1 - Z_2\| \}.$$

__Thm__. The space $\mathrm{Stab}_\Lambda(\mathcal{D})$ of stability condition is a smooth finite-dimensional complex manifold s.t

$$\mathcal{Z} : \mathrm{Stab}_\Lambda(\mathcal{D}) \longrightarrow \mathrm{Hom}(\Lambda, \mathbb{C})$$
$$\sigma = (Z, P) \longmapsto Z$$

is a local homeomorphism at every point of $\mathrm{Stab}_\Lambda(\mathcal{D})$.

$GL^+(2,\mathbb{R}) \rightsquigarrow \widetilde{GL}^+(2,\mathbb{R}) = \{ (T, f) \mid$ $T : \mathbb{R}^2 \to \mathbb{R}^2$ orientation preserving linear isomorphism

$f : \mathbb{R} \to \mathbb{R}$ increasing map

$f(\phi+1) = f(\phi) + 1$

such that the induced map

$\mathbb{R}^2 \setminus \{0\} \big/ \mathbb{R}_{>0} \simeq S^1 = \mathbb{R} \big/ 2\mathbb{Z}$ are the same $\}$

__e.g.__ id can be lifted to $(id, f : \phi \to \phi + 2k) \in \widetilde{GL}^+(2,\mathbb{R})$.

$k \in \mathbb{Z}$

$$\mathrm{Stab}_\Lambda(\mathcal{D}) \times \widetilde{GL}^+(2,\mathbb{R}) \longrightarrow \mathrm{Stab}_\Lambda(\mathcal{D})$$

$$\big(\sigma = (Z,P), \quad (T,f)\big) \longmapsto \sigma' = (Z', P')$$

$$Z' = T^{-1} \circ Z, \quad P'(\phi) = P(f(\phi))$$

* semistable objects w.r.t stability condition σ and σ' are the same.
but only the phases have been relabeled.

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stability condition on curves of $g > 0$

$\Lambda := K_{num}(D^b(C))$ ⇝ we call them numerical stability condition.

smooth projective curve of $g>0$

<u>thm</u>. $\mathrm{Stab}_\Lambda(D^b(C))$ consists of exactly one $\widetilde{GL}(2,\mathbb{R})$-orbit

i.e. $\mathrm{Stab}_\Lambda(C) / \widetilde{GL}^+(2,\mathbb{R}) = \{pt\}$.