

$(X, H)$  : polarized K3 surface with  $\text{Pic}(X) = \mathbb{Z} \cdot H$ .

set  $B = \underline{b}H$ ,  $w = aH$ ,  $b, a \in \mathbb{R}$ ,  $\underline{a} > 0$ .  $\leadsto \text{Ch}^{w, B}(X) =: \text{Ch}^b(X)$ .

$$\tilde{Z}_{w, B}(E) / H^2 = i \left( \underline{a} \left( \frac{ch_1 \cdot H}{H^2} - r b \right) + r \left( -\frac{b^2}{2} + \frac{a^2}{2} - \frac{1}{1+2} \right) + b \frac{ch_1 \cdot H}{H^2} - \frac{ch_2}{H^2} \right)$$

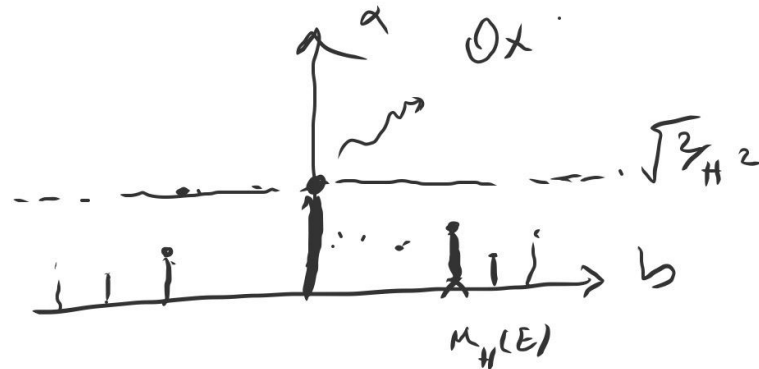
$$ch(E) = (r, ch_1, ch_2)$$

$$\tilde{Z}_{a, b}(E) := i \left( \frac{ch_1 \cdot H}{H^2} - r b \right) + \text{---}''\text{---}$$

Cor.  $\sigma_{a, b} = (\text{Ch}^b(X), \tilde{Z}_{a, b})$  is a stability condition on  $X$ .

iff there is no spherical sheaf  $E$  of rank  $r$  with  $\mu_H(E) = b$  and

$$a \leq \frac{1}{r} \sqrt{2} H^2$$



Wall and chamber decomposition for a fixed object  $\underline{E} \in D^b(X)$ :

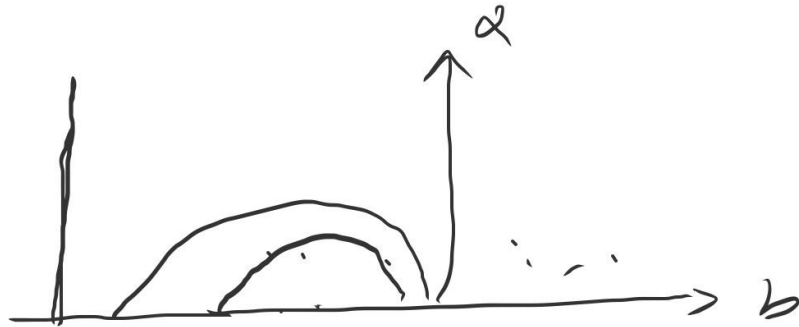
the space of numerical stability conditions admits a chamber decomposition, depending on  $\underline{E}$ , s.t.

- for a chamber  $C$ ,  $\sigma$ -stability or instability of  $\underline{E}$  is independent of the choice of  $\sigma \in C$ .
- Walls consists of stability conditions when  $\underline{E}$  is strictly semistable and  $\underline{E}$  is unstable for stability conditions in one of the adjacent chambers to the wall.

In  $(a,b)$ -plane, the walls are either semicircles with center on the  $b$ -axis

\*  $\rightsquigarrow$  or vertical lines.

\* the walls for  $\underline{E}$  cannot intersect.



$$\frac{-\operatorname{Re}[Z_{a,b}(\underline{E})]}{\operatorname{Im}[Z_{a,b}(\underline{E})]} = \frac{-\operatorname{Re}[Z_{a,b}(\underline{F})]}{\operatorname{Im}[Z_{a,b}(\underline{F})]}$$

Q. Can we uniquely reconstruct a K3 surface from a curve on the surface?

$F_g$ : moduli space of polarized K3 surfaces  $(X, H)$  where  $H$  is a primitive ample line bundle on  $X$  with  $H^2 = 2g - 2$ .

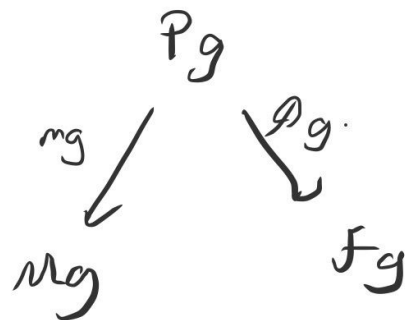
$\hookrightarrow \dim F_g = 19$

$P_g$ : moduli of triples  $(X, H, C)$ ,  $(X, H) \in F_g$ ,  $C \in |H|$  smooth curve.

$\hookrightarrow \dim P_g = 19 + g$

$M_g$ : moduli space of smooth curves of genus  $g$

$\hookrightarrow \dim M_g = 3g - 3$



[Ciliberto - Lopez Mirand] if  $g \geq 11$ ,  $g \neq 12$ , the map  $m_g$  is birational onto its image.  $3g - 3 \geq 19 + g$   
 $\hookrightarrow \underline{g \geq 22}$ .

In 2001, Mukai introduced a program to find the rational inverse of  $mg$ ,  
where  $g = 2s+1$ ,  $s \geq 5$  odd.

① let  $C$  be a general curve in the image of  $mg$ .

$$BN_C(2, 4s, s+2) = \left\{ \begin{array}{l|l} \text{stable rank 2} & \deg(E) = 4s \\ \text{vector bundles} & h^0(E) \geq s+2 \end{array} \right\} \text{ on } C$$

②  $BN := BN_C(2, 4s, s+2)$  is a K3 surface.

③ let  $E$  be a universal sheaf on  $C \times BN$ , and let  $H'$  be a generic polarization on  $BN$ ; let  $v' = v(E|_{P \times BN})$  for a point  $P \in C$ .

The K3 surface  $X$  which contains  $C$  is isomorphic to  $M_{BN, H'}(v')$  of  $M_H$ -stable sheaves on  $BN$  with Mukai vector  $v'$ .

$M_{X, H}(v)$  = moduli of  $M_H$ -stable sheaves on  $X$  with Mukai vector  $v = (2, H, B)$   
 $\leadsto \dim M_{X, H}(v) = \sqrt{v^2} + 2 = 2 \leadsto M_{X, H}(v)$  is a K3 surface

$$\begin{aligned} \varphi: M_{X,H}(\underline{v}) &\longrightarrow BN_C(\underline{2, 4s, s+2}) & C \in |H| \\ E &\longrightarrow E|_C & (*). \end{aligned}$$

It we prove  $\varphi$  is well-defined and an isomorphism, then Mukai's program works. (\*)

step 3 follows by duality of polarised K3 surfaces,

$$X \rightsquigarrow M_{X,H}(\underline{v}) \cong BN \rightsquigarrow M_{BN,H'}(\underline{v}^*) \simeq X$$

Remark. Mukai's program can be extended to any  $g \geq 11$ ,  $g \neq 12$  [Fey27].

Let  $(X, H)$ ,  $\text{Pic}(X) = \mathbb{Z} \cdot H$ ,  $C \in |H|$ ,  $H^2 = 2g - 2 = 4s$ .

Claim.  $\varphi$  is well-defined.

Take an  $M_H$ -stable sheaf  $E \in M_{X,H}(\underline{v})$   $\underline{v} = (2, H, s)$

$$\rightsquigarrow E^{**} \text{ is } M_H\text{-stable} \rightsquigarrow \chi(E^{**})^2 \geq -2$$

$$E \hookrightarrow E^{**} \rightarrow Q.$$

$\downarrow$   
 torsion sheaf  
 $v(Q) = (0, 0, 9)$

$v(E^{**}) =$   
 $(2, 11, 8+9)$

$$v(E^{**})^2 = 45 - 4(8+9) = -49 \geq -2$$

$$\Rightarrow 9 = 0$$

$$\Rightarrow E = E^{**} \quad \text{so } E \text{ is a vector bundle on } X.$$

Ex.  $E$  and  $E(-H)[1] \in \text{Coh}^p(X)$  and  $\sigma_{(\alpha, 0)}$ -stable for  $\alpha \gg 0$

\*  $\sigma_{\alpha, 0}$  is a stability condition for any  $\alpha > \sqrt{2}_{H^2}$ .

$\text{Im}[Z_{\alpha, 0}(E)] = 1$  is minimal, then  $E$  is  $\sigma_{\alpha, 0}$ -stable for any  $\alpha > \sqrt{2}_{H^2}$ .  
 this same is valid for  $E(-H)[1]$ .

$$\exists \alpha_0 > \sqrt{2}_{H^2} \text{ s.t. } \phi_{(\alpha_0, 0)}(E) = \phi_{\alpha_0, 0}(E(-H)[1])$$

$$\underbrace{E \hookrightarrow E^* E \rightarrow E(-H)[1]} \text{ in } \text{Coh}^p(X) \text{ so is } E|_C \text{ is } \sigma_{\alpha_0, 0}\text{-semistable}$$

$$\Rightarrow \underbrace{E|_C} \text{ is } \sigma_{\alpha, 0}\text{-stable for } \alpha_0 < \alpha < \alpha_0 + \varepsilon.$$

$$\begin{aligned} \alpha(E) &= 8+2 \\ &= h^0(E) - h^1(E) \end{aligned}$$

$$\Rightarrow E|_C \text{ is a slope-stable vector bundle.} \Rightarrow h^0(E) \geq 8+2$$

$$\Rightarrow \underbrace{h^0(E|_C)} \geq 8+2.$$

claim.  $\varphi: \underline{M_{X,H}(u)} \rightarrow \underline{BN_C(2, 4s, s+2)}$  is bijective.

①  $F \in BN_C(2, 4s, s+2) \rightsquigarrow \text{if } F \in \text{coh}^0(X)$ .

claim. Since  $\text{if } F$  is  $H$ -Gieseler-stable,  $\text{if } F$  is  $G_{a,b}$ -stable for  $a > 0, b \in \mathbb{R}$ .

② The first wall for objects of class  $v(\text{if } F)$  is made by  $E \hookrightarrow \text{if } F \rightarrow E(-H) \otimes \mathbb{I}$ . (\*)

$$\Rightarrow F \cong E|_C.$$

Since (\*) is the HN filtration of  $E|_C$  below the wall  $w$ ,  
 uniqueness of HN filtration implies that for any  
 short  $E|_C \in BN$ , there exists a unique  $E \in M_{X,H}(u)$  in  
 the inverse image  $\Rightarrow \varphi$  is injective.



BN Lemma. Take a pair  $(a, b)$  sufficiently close to  $(\sqrt{\frac{2}{H^2}}, 0)$  s.t.  $b < 0$ ; let  $G \in D(X)$  be a  $\sigma_{a,b}$ -stable object with the same phase as the structure sheaf  $\mathcal{O}_X$ ; then.

$$h^0(X, G) \leq \frac{\chi(G)}{2} + \frac{\sqrt{(\chi_2(G))^2 + (2H^2 + 4) \left( \frac{\chi_1(G) \cdot H}{H^2} \right)^2}}{2}$$

Pf.  $ev: \text{Hom}(\mathcal{O}_X, G) \oplus \mathcal{O}_X \rightarrow G \rightsquigarrow \underbrace{ev} \text{ in } P(\mathcal{O})_{(1,b)}$

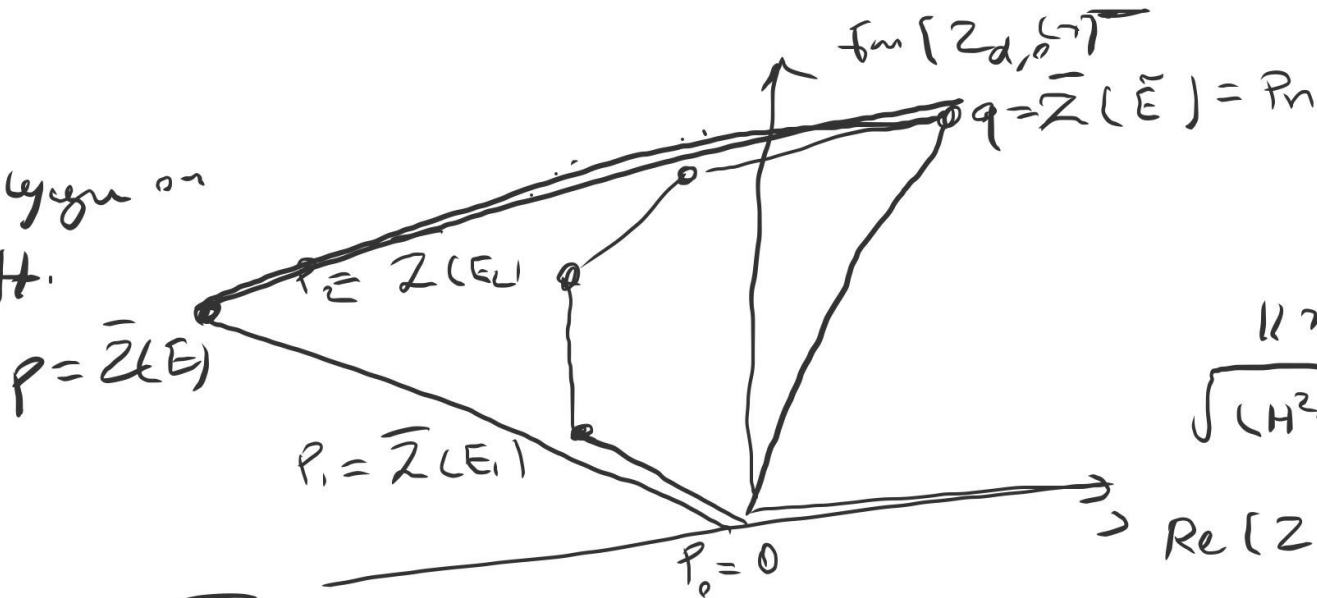
$$\Rightarrow \sqrt{(\underbrace{ev}_{\text{in } P(\mathcal{O})_{(1,b)}})^2} \geq -2 \left( \frac{\chi_1(G) \cdot H}{H^2} \right)^2.$$

(4) Assume that  $\gamma$  is not surjective,  $\exists F \in BN_C$  s.t.  $F \notin E|_C$ ,  $E \in M_{X,H}(\gamma)$

рнур

Given an object  $E \in \text{Coh}^p(X)$  which has no subobject  $E' \subset E$  in  $\text{Coh}^p(X)$  with  $\dim(E') = 0$ , there exist  $\alpha^* > \sqrt{\frac{2}{H^2}}$  s.t. the HN filtration of  $E$  is a fixed sequence  $0 = E_0 \subset E_1 \subset \dots \subset E_n = E$  for all stability conditions  $\sigma_{d,p}$  with  $\sqrt{\frac{2}{H^2}} < \alpha < \alpha^*$ .

HN polygen on  
ten left.



$$\|x+iy\| = \sqrt{(H^2x)^2 + (2H^2+4)y^2}$$

behe  $\alpha \rightarrow \sqrt{\frac{3}{4}}$

$$\lim_{\alpha \rightarrow 0} \bar{Z}_{d,0}(E_i) = - \frac{dh_2(E_i)}{H^2} + i \frac{dh_1(E_i) \cdot H}{H^2} =: \bar{Z}(E_i)$$

We can apply BN lemma for cell  $E_i/E_{i-1}$ .

$$h^0(E_i/E_{i-1}) \leq \frac{\chi(E_i/E_{i-1})}{2} + \frac{\sqrt{(\chi_2(-))^2 + (2H^2+4) \left( \frac{\chi_1(-) \cdot H}{H^2} \right)^2}}{2}^*$$

$$= \frac{\chi(E_i/E_{i-1})}{2} + \frac{\| \overline{P_i P_{i-1}} \|}{2}$$

$$h^0(\tilde{E}) \leq \sum h^0(E_i/E_{i-1}) \leq \frac{\chi(\tilde{E})}{2} + \frac{1}{2} \sum_{i=1}^n \| \overline{P_i P_{i-1}} \|$$

$F \in BNC$

$$s+2 \leq h^*(i_\alpha F) \leq \frac{1}{2} \sum_{i=1}^n \| \overline{P_i P_{i-1}} \|$$

$$\leq \frac{1}{2} (\| \overline{OP} \| + \| \overline{PQ} \|) = s+2$$

↙

$\Rightarrow \psi$  is surjective.