Bonus facts / clarifications :

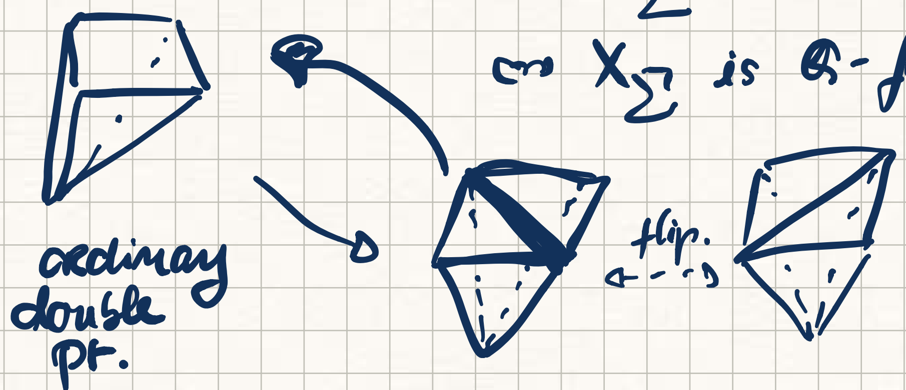
1. The orbit of the origin is $T_N$

2. $X_\Sigma$ • simplicial = all cone generators are linearly independent

$\Leftrightarrow X_\Sigma$ orbifold $\Leftrightarrow$

$\Leftrightarrow X_\Sigma$ is $\mathbb{Q}$-factorial.

NOT :




- has isolated sing : all faces of $P$ have length 1 edges.

- is terminal : only lattice pts in $P$ are its vertices & the origin.

- canonical : No strictly interior lattice points except for the origin.

- Gorenstein: all faces of $P$ are at height 1.

- all blow-ups can be seen as subdivisions of the fan.

# Polytopes in M.

Let $X_\Sigma$ a projective toric variety with ray generators $u_1 \dots u_r \in \mathbb{Z}^n = N$.

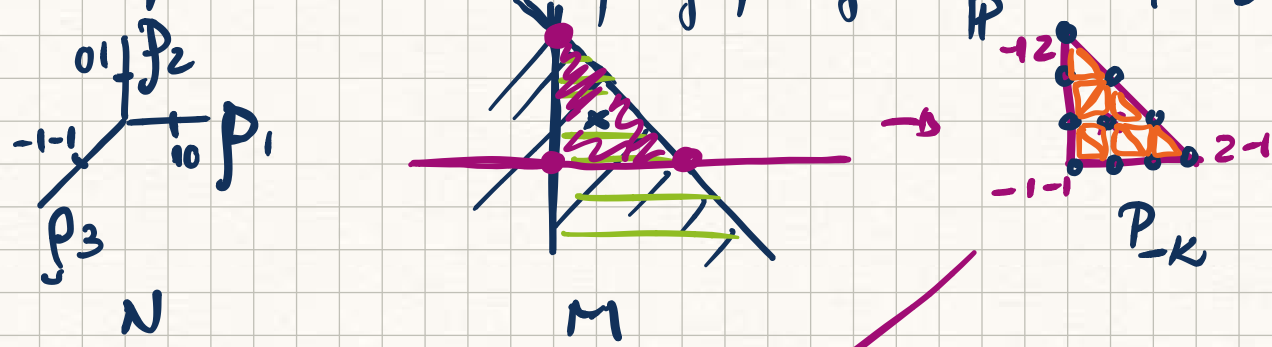
Let $D = d_1 D_1 + \dots + d_r D_r$ a torus-inv. divisor.

One can associate a (possibly empty) polytope to any such divisor:

$$P_D = \{ m \in M_{\mathbb{R}} \mid \underline{\langle m, u_i \rangle + d_i \geq 0} \ \forall i = 1, \dots r \}.$$

- if $X_\Sigma$ is smooth, this always has lattice vertices.
- its lattice pts represent the sections of $D$.

**Examples:** Build the polytope of $-K_{\mathbb{P}^2} = D_1 + D_2 + D_3$



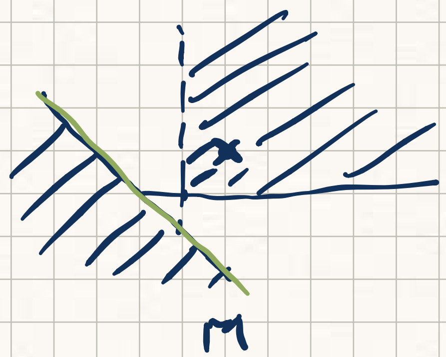

tells us several things:

- there are $\underbrace{10}$ sections in $H^0(-K)$
- the degree $(-K)^2 = \text{volume}(P_K) = 9$.
- $-K$ is divisible by 3 in $Cl(\mathbb{P}^2)$ (bc $P_{-K}$ is a 3-dilate of $\underline{\underline{\Delta = \mathcal{O}(1)}}$)

Rk: all polytopes of divisors in the same lin system look the same.

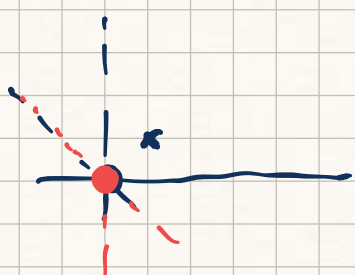
- the polytope is $\neq \emptyset$ if $D$ is nef.
- the polytope is full-dimensional if $D$ is big.

$\to$ $P_D$ can be empty, for ex: $P_{D_1+D_2-3D_3}$:



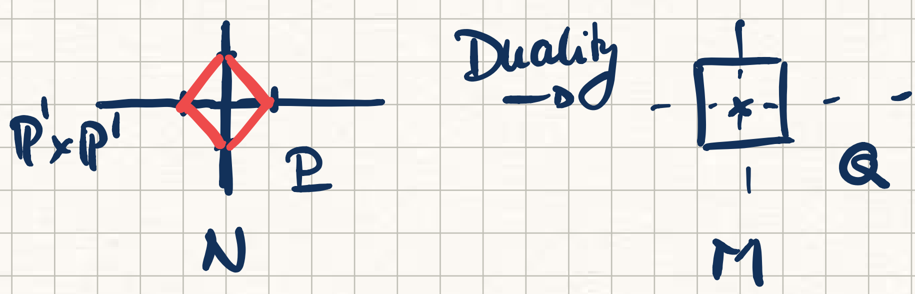

$\Rightarrow$ $D_1+D_2-3D_3$ is not nef.

$\to$ $P_D$ can be a point: $P_{D_1+D_2-2D_3}$

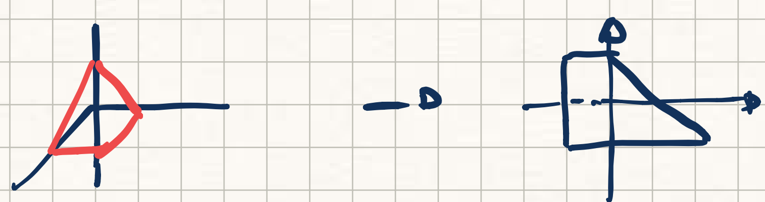


This is nef, but not big.

other fans: $P_{-K}$ for $\mathbb{F}_0$, $\mathbb{F}_1$, $\mathbb{F}_a$




$F_1$



Home work:

write $P_{-K}$ for

$F_2$

! $P_{-K}$ has as many facets as the rays in the fan

no longer true.

**Remark:** To each character $m \in M$ we associate a unique $\mathbb{Z}$- combination of the $\{D_i\}_1^r$

So we have a group monomorphism between lattices

$$0 \to M \longrightarrow \bigoplus_{\rho_i \text{ ray}} \mathbb{Z} D_i = \mathrm{Div}_{T_N} X_\Sigma \simeq \mathbb{Z}^r \xrightarrow[\text{lin. equivalence.}]{} Cl\, X_\Sigma \to 0. \quad (*)$$

This is part of a bigger picture of building $X_\Sigma$ as a GIT quotient.

**Question:** → Cones are not natural. Quotients by tori are natural.

Baby: $\lambda \cdot (\underbrace{x_0 \dots x_n}) = (\lambda x_0 \,\text{—}\, \lambda x_n)$

$\mathbb{P}(\underline{1\,1\,1 \dots 1})$

$$\mathbb{C}^{n+1} \setminus \{0\} / \mathbb{C}^*$$

have to remove origin to get correct quotient.

Toddler: wPS $\mathbb{P}(a_0 \dots a_n)$

$$\lambda \cdot (x_0 \,\text{—}\, x_n) = (\lambda^{a_0} x_0 \,\text{—}\, \lambda^{a_n} x_n)$$

$$\mathbb{C}^{n+1} \setminus \{0\} / \underset{\sim}{\mathbb{C}^*}$$

**Wait:** $(\mathbb{C}^*)^k$ act on $\mathbb{C}^{\text{something}}$.

Will need to remove the vanishing locus of some ideal.

- will organise the actions in a <u>weight matrix</u>.

$$
\begin{array}{c c}
 & \begin{matrix} x_0 & \cdots & x_n \end{matrix} \\
k \left\{ \right. & \begin{matrix} a_{10} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{k0} & \cdots & a_{kn} \end{matrix}
\end{array}
$$

$\longrightarrow$ this matrix has to be "nice"

- no generic stabilisers
- no stabilisers along divisors.

Cox coordinates

will build $X_\Sigma = \mathbb{C}^{n+1} /\!\!/ (\mathbb{C}^*)^k$

$\triangleright$ similarly to WPS there is a formula for computing $-K = \begin{pmatrix} \sum a_{1i} \\ \vdots \\ \sum a_{ki} \end{pmatrix}$